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FORMULATION, COMPUTATION, AND SOLUTION PROCEDURES FOR MATERIAL AND/OR GEOMETRIC NONLINEAR STRUCTURAL ANALYSIS BY THE FINITE ELEMENT METHOD

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FORMULATION, COMPUTATION, AND SOLUTION PROCEDURES FOR MATERIAL AND/OR GEOMETRIC NONLINEAR STRUCTURAL ANALYSIS BY THE FINITE ELEMENT METHOD

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FOREWORD

The results presented in this report were obtained under Sandia Contract 82-5617. This research was partially funded under NASA Grant NGL-44-001-044 as a result of interest in the thermal cycle fatigue problems as related to the Space Shuttle. Appreciation is expressed to Dr. Fred Stebbins of the Manned Spacecraft Center for constructive criticism.

Two papers have been accepted which relate to different aspects of the research reported herein. The first emphasizes the formulation and computational procedures and was presented at the National Symposium on Computerized Structural Analysis and Design held at George Washington University, Washington, D. C. on March 27-29, 1972. The second paper summarizes the evaluation of the solution procedures and was presented at the AIAA/ASME/SAE 13th Structures, Structural Dynamics, and Materials Conference held in San Antonio, Texas, on April 10-14, 1972.

ABSTRACT

In this report computational and solution procedures are explored for geometric and/or material nonlinearities. A highly efficient computational procedure based on a combined finite element-finite difference approach is developed. Several different solution procedures are evaluated and particular solution procedures are recommended for geometric, material, and combined geometric-material nonlinearities. It is concluded that, through a proper choice of computational and solution procedures, nonlinear problems may be solved in reasonable times on the computer. It is also concluded that the development of computer codes for large scale problems is feasible and potentially economical.

SIGNIFICANT CONTRIBUTIONS

The original purpose of this research was to evaluate the efficiency of various solution procedures for material nonlinearities and for combined material-geometric nonlinearities. However, it was found during the course of this investigation that more efficient computational procedures were needed for computing the plastic strains, pseudo forces, and tangent stiffness matrices. Advances in these areas in addition to the evaluation of solution procedures are considered to be significant. In summary the significant contributions are:

1. A literature survey is given which summarizes the contributions of other researchers for material nonlinearities and combined geometric-material nonlinearities.

2. The development of a computational procedure for stresses and plastic strains which stays on the assumed uniaxial stress-strain curve regardless of the size of the load increment.

3. A formulation of all solution procedures from equations of equilibrium in terms of pseudo forces for all nonlinear terms.

4. The formulation of Item 2 leads directly to a computional procedure which is highly efficient. Computation of the tangent stiffness matrix for combined geometric-material nonlinearities requires very little computational effort over that required to compute the pseudo forces. The computational procedure is an order of magnitude faster than the procedures quoted by other researchers.

5. Solution procedures are evaluated in terms of accuracy, usability,

and computational effort. Plastic loading combined with elastic unloading is used in the evaluation.

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NOMENCLATURE

⌊ ⌋	= row matrix
{ }	= column matrix
[ ]	= square matrix
A	= area of element
B	= matrix relating strains to generalized displacements and scalar defined by Eq. 66.
C	= matrix relating increment of plastic strain to increment of total strain, also scalar coefficient in Eq. 13.
D	= matrix relating stress to elastic strain
E	= Young's Modulus
H'	= scalar relating increment of uniaxial stress to increment of uniaxial strain
K	= matrix of stiffness coefficients
L	= meridional length of element
M	= number of subincrements used in computing plastic strain increments
P	= load parameter (P_{max} = 100)
$\bar{P}$	= normalized load vector
Q	= psuedo force due to nonlinearities
V	= undeformed volume
W	= weighting factor for numerical integration
Z	= scalar coefficient in Eq. 13
e	= linear expression for mid-surface strains
f	= force unbalance in equation of equilibrium (Eq. 2)

q = generalized displacements

r = radial coordinate for mid-surface of shell

t = thickness of shell

u = displacement in meridional direction

w = displacement normal to shell

z = distance from mid-surface of shell

Δ = increment

ε = strain

$\bar{\varepsilon}$ = equivalent uniaxial value of strain

θ = circumferential angle

X = changes in curvature

ν = Poisson's ratio

σ = stress

ϕ = slope of shell element (Fig. 3)

Superscripts

$\cdot$ = differentiation with respect to load parameter P.

A = area

e = elastic contribution

NL = contribution due to geometric nonlinearities

P = contribution due to material plasticity

T = thermal contribution

z = pertaining to factors that depend on the distance from the mid-surface

Subscripts

i = load increment, degree of freedom, dummy summation variable

j = summation index

L	=	linear contribution
NL	=	nonlinear contribution
s	=	meridional direction
θ	=	circumferential direction

INTRODUCTION

The finite element method of structural analysis has advanced quite rapidly on many fronts in recent years. One such area is in nonlinear analyses where the nonlinearities are due to large rotations and/or plastic deformations. Advancements in this field have led to more efficient designs ranging from pressure vessels to thermal protection systems for space vehicles where thermal fatigue is quite important.

With the increasing interest in nonlinear analysis, many investigators have realized that most of the present procedures for conducting nonlinear analyses for large systems involving many degrees of freedom are very inefficient and require excessive amounts of computer time. This is especially true for nonlinear analysis by the finite element method. In fact, some proponents of the finite difference procedures have stated that the finite element method is not economical for nonlinear analyses.

The long computer run times may be traced to several different sources.

1. Formulation: The majority of the computer codes for the geometric nonlinear analyses use the Eulerian formulation which requires that the nonlinear stiffness matrix must be transformed from the deformed position to the global coordinates. This transformation is quite time consuming. In addition, some codes include nonlinearities in the curvature as well as the mid-surface strains, which may or may not be needed.

2. Excessive use of matrices: Many researchers describe needed quantities as a long series of matrix multiplications, even when many of the matrices contain only a few non-zero terms. While matrix notation serves as an effective mathematical tool, many of the multiplications should be performed as summations over only the non-zero variables.

3. Solution procedure for nonlinear problems: By far the most popular procedure for the solution of problems in nonlinear structural mechanics is the incremental approach. For geometric nonlinear problems the incremental approach is very inefficient. However it is quite easy to use which probably accounts for its popularity.

4. Approximations for nonlinear terms: Some researchers prefer to use the same displacement function for the evaluation of the nonlinear terms as was used to obtain the stiffness matrix. For some problems this may require numerical integration of a large number of terms for each load step and thus is very time consuming. In general, the degree of approximation which should be used depends on the solution procedure. For example, it does not seem justifiable or economical to make crude approximations for the nonlinear terms if an incremental approach is being used where the majority of the computer time is expended in the solution of the equations. The above example is counter to the procedures developed herein.

Computational Philosophy

To better interpret the contents of this report it is deemed desirable to explain the philosophy adopted at Texas A&M. First it is believed that no effort should be spared in computing the element stiffness matrix. For shells of revolution the authors and their colleagues[78] use curved elements, higher order displacement functions, and static condensation to obtain the element stiffness matrix. Secondly, many simplifying approximations are used in the treatment of nonlinearities. For geometric nonlinearities this consists of using conical frustum elements and linear functions for all displacements. This degree of approximation is justified by the fact that conical frustum elements have proven to be useful in computing the midsurface strains and rotations. As the geometric nonlinear terms depend only on the midsurface strains and rotations, they are computed to a sufficient degree of accuracy. The effects of plastic deformations are computed by assuming that the plastic strains are constant over the length of the element but vary through the thickness. All strains in the nonlinear terms are written as finite difference expressions in terms of the nodal displacements.

The simplifying assumptions for the nonlinear analysis of shells of revolution are directed somewhat by the narrow band width of the stiffness matrix. Very little computer time is required to solve the set of equations and, thus, it has been found more efficient to use the simplifying assumptions for the nonlinear terms and a large number of elements than to use a few complex elements. This philosophy has led to the development of computer codes which yield accurate solutions with computer time requirements competitive with any other code.

For systems where the narrow band width does not exist the authors and their colleagues[119,76] have worked toward the development of solution procedures which require a single inversion of the stiffness matrix. Some success has been achieved in this area as indicated by a recent publication on this subject.[120] Once these solution procedures are perfected the authors believe that complex problems can best be treated by making reasonable simplifying assumptions for the effects of nonlinearities.

Literature Survey

The literature survey was conducted by studying numerous papers and making copies of the papers which appealed to the authors. Undoubtedly, the literature survey is incomplete and it will be appreciated if other papers are called to our attention. Further, many of the papers listed in the references will not be discussed in this section. The primary criterion for discussing the paper is whether or not it contributes something new to the formulation or method of solution for plasticity problems.

To start the survey it may be valuable to point out other survey papers which contribute significantly to the field. First, in the fundamental theory of plasticity, a very comprehensive review of the various available theories is given by Isakson, Armen, and Pifko[47] and less complete, but in a well written manner by Khojasteh-Bakht[50] and Felippa.[25] Interestingly none of these references cover the sublayer model used so successfully by the finite difference researchers. Discussions of this model may be found in Refs. 44, 54, and 127.

The present trend is to use the Von Mises yield criteria and associated flow rule. Isotropic hardening is frequently used but it can be expected that in the future more use will be made of Ziegler's modification of Prager's hardening rule[132] as it properly accounts for the Bauschinger effect. The present research uses the Von Mises yield criteria, isotropic hardening, and elastic unloading.

Survey papers pertaining to geometric nonlinearities include those by the late Harold Martin,[74,75] Oden,[85] and several from the group at

Texas A&M.[34,35,119] References 119 and 23 show the equivalence between the Eulerian and Lagrangian formulation for a simple truss element while Refs. 34 and 35 present rather complete literature surveys for geometric nonlinearities and a comparison of solution procedures. The literature on geometrically nonlinear problems will not be covered in the present survey. This survey is limited to the solution of plasticity problems and for combined material-geometric nonlinearities. The survey is also limited to small strains.

The survey of the literature will be presented according to the method used in the solution of the resulting equations of equilibrium. For the present, the only type of nonlinearities considered are those due to plastic strains. The basic equations of equilibrium are of the form

$$[K]\{q\} = \{P'\} + \{Q(q)\} \tag{1}$$

where $[K]$ = global stiffness matrix

$\{q\}$ = column matrix of generalized displacements

$\{P'\}$ = generalized forces due to applied loads

$\{Q(q)\}$ = column matrix of pseudo forces due to plastic (initial) strains.

Ideally, Eq. 1 is exactly satisfied. However, with many methods, the equations of equilibrium are only satisfied to within a certain degree of accuracy. For this reason it is convenient to define an unbalance of force term given by

$$\{f\} = -[K]\{q\} + \{P'\} + \{Q\} \tag{2}$$

where $\{f\}$ is the amount of out of balance in force which exists.

The first class of solution procedures are those which satisfy the equations of equilibrium exactly. For this class, $\{f\} = \{0\}$.

The first procedure used to satisfy $\{f\} = \{0\}$ is the method of successive approximations. In this method the load is increased in increments and, at each increment of load, iteration until covergence is performed using the recursion relation

$$[K]\{q_i\} = \{P_i'\} + \{Q_{i-1}\} \quad (3)$$

where i is the ith iteration and $\{Q_{i-1}\}$ are the pseudo plastic forces based on the generalized displacements at the (i-1)th iteration. The process is usually started by a simple linear extrapolation based on previous solutions. Mendelson and Manson[79] first proposed the method of successive approximations in 1959. Others to discuss the method are Argyris,[3] De Donato,[17] Fowler,[26] Havner,[36] Marcal,[67] and Witmer and Kotanchik.[128] The main difficulties which are encountered with the method are the very slow rate of convergence for large plastic strains and the tendency to converge to the incorrect answer for elastic-perfectly plastic materials. De Donato[17] shows that his iterational procedure fails for perfectly plastic materials. Havner[36] stipulates a monotonically increasing stress strain curve and shows that the iterational procedure will converge but does not show the rate of convergence. Marcal[67] shows that the method of successive approximations converges to the incorrect answer for elastic-perfectly plastic materials. Witmer and Kotanchik[128] use this method to solve complex shell of revolution problems and state in their conclusions that some technique is needed to accelerate convergence.

Zienkiewicz, Valliappan, and King[134] have presented a method of successive approximations which they call the initial stress approach. In Ref. 135 the authors show that the method of initial stress is in essence the same as the method of successive approximation.

Another solution procedure of the class $\{f\} = \{0\}$ is the Newton-Raphson procedure first used by Oden and Kubitza.[87] They solved for the deflection of a square plate with external pressure. The only type of stess-strain behavior considered was elastic-plastic with appreciable strain hardening without unloading. It will be shown in the present research that consideration of elastic unloading causes the Newton-Raphson procedure to fail to converge in many cases.

Another solution prodecure which satisfies the equation of equilibrium exactly is the minimization of the total potential energy as presented by Stanton and Schmit.[114] In Ref. 114 the authors use deformation theory instead of the more accurate and commonly used incremental theory. This particular solution procedure is not evaluated in the present study.

A second class of solution procedures and by far the most popular to date contains those methods which solve the equation $\{\dot{f}\} = \{0\}$ where the dot indicates differentiation with respect to a load parameter. For example, the load vector $\{P'\}$ may be written as

$$\{P'\} = P\{\bar{P}\} \tag{4}$$

where P is some convenient normalizing term which, for convenience, is taken as ranging from 1 to 100. Differentiation is then taken with respect to P. From Eq. 2 the expression for $\{\dot{f}\} = \{0\}$ becomes

$$[K]\{\dot{q}\} = \{\overline{P}\} + \{\dot{Q}\} \tag{5}$$

A frequently used solution procedure for Eq. 5 is to use an Euler forward difference for $\{\dot{q}\}$ and a backwards difference for $\{\dot{Q}\}$. Thus, for any increment

$$[K]\{\Delta q\}_i = \Delta P\{\overline{P}\} + \{\Delta Q\}_{i-1} \tag{6}$$

It is noted that Eq. 6 has a truncation error of ΔP.

Another form which is equivalent to Eq. 6 is in terms of the total displacement and total pseudo plastic forces

$$[K]\{q_i\} = \{P_i'\} + \{Q_{i-1}\} \tag{7}$$

The form given by Eq. 7 is obtained by expanding $\{\dot{Q}_i\}$ in a Taylor's series expansion and maintaining only the constant term.

$$\{\dot{Q}_i\} = \{Q_{i-1}\} + \Delta P\ \{\dot{Q}_{i-1}\} \tag{8}$$

It is seen that Eq. 7 has an error term of ΔP. Equation 6 and 7 are essentially the same except that one is written in terms of increments of displacements and one is written in terms of total displacements. This is easily seen by noting that Eq. 6 may be obtained by subtracting Eq. 7 at i-1 from Eq. 7 at i.

Finally, returning to Eq. 5, chain rule differentation may be used on the pseudo force term.

$$\{\dot{Q}\} = [\frac{\partial Q_i}{\partial q_j}]\{\dot{q}\} \tag{9}$$

Substituting Eq. 9 into Eq. 5 results in

$$[[K] - [\frac{\partial Q_i}{\partial q_j}]]\{\dot{q}\} = \{\dot{\bar{P}}\} \quad (10)$$

Equation 10 may be solved by any of a large number of numerical procedures. However the most common procedure is a simple forward integration procedure.

The matrix $[\frac{\partial Q_i}{\partial q_j}]$ in Eq. 10, although derived by a different approach, is referred to here as the contribution to the tangent stiffness matrix. It will be shown in the section on formulation that the combination of the plasticity stiffness matrix and the usual linear stiffness matrix is exactly the same as obtained from the incremental form of the virtual work expression. In much of what follows, the plasticity stiffness matrix will be denoted as

$$[K^P] = - [\frac{\partial Q_i}{\partial q_j}] \quad (11)$$

Gallagher, Padlog and Bijlaard[28] used Eq. 7 to solve plasticity problems as early as 1962. In Ref. 28 they presented two approaches for computing the plastic strains. These methods are referred to as the constant stress and constant strain approaches. It was shown that the constant stress method encounters numerical instabilities. These numerical instabilities can not be overcome by reducing the load increment. In fact, the reverse occurs in that the smaller the load increment, the sooner the numerical instabilities occur. This approach or its equivalent given by Eq. 6 has also been pursued in Refs. 3,5,46,56 and 82. The general consensus of opinion gained from these references is that the initial strain approach given by Eqs. 6 or 7 is very slow to converge with reduction in the size of the load increment.

Isakson, Armen, and Pifko[47] present a slight modification of the solution procedure given by Eq. 7. In their method, called the predictor method, the pseudo plastic force terms are estimated based on their values at previous loads. For example, this may be accomplished by a simple linear extrapolation. The use of an extrapolation procedure does not seem to introduce numerical instabilities into the solution procedure. For geometric nonlinearities, on the other hand, solutions by Eq. 7 become unstable for moderate nonlinearities and any type of extrapolation procedure tends to hasten the instability.[34]

The incremental procedure as given by a simple forward difference solution of Eq. 10 was developed by Pope,[95] Swedlow and Yang,[121] Marcal and King[72] and Yamada.[131] The incremental procedure was further developed by Armen et al.[5] where Ziegler's modification of Prager's hardening rule was used and problems involving cyclic loading were solved. In Ref. 5 the shear lag problem was solved by both the initial strain and the incremental method. Results show that the incremental procedure converges quite rapidly compared with the initial strain procedure. Further comparisons are given in Refs. 50 and 51 with the incremental tangent stiffness method being judged the most useful.

Felippa[25] solved Eq. 10 by a two step method where the tangent stiffness matrix is evaluated one half increment forward using a simple forward difference procedure. This is sometimes referred to as a chord stiffness matrix. Reference 25 also implies that the problem may be formulated in terms of a first order nonlinear differential equation. Additional discussion of the half step formula is presented by Akyuz and Merwin.[1]

Blacklock[11] and Richard and Blacklock[104] solve Eq. 10 by the Runge-Kutta procedure and present an inverse Ramberg-Osgood curve for the uniaxial stress-strain relation. Results presented in Ref. 104 show that excellent results may be obtained using a small number of increments with the Runge-Kutta solution procedure. However, it should be pointed out that the equations must be solved four times for each increment of load and hence, this procedure may be quite time consuming.

The advantage of writing the equilibrium equations in the form given by Eq. 10 is that this form opens the door to a very large class of solution techniques. These include simple forward differences, predictor-corrector schemes, Runge-Kutta methods, and so on. However, it has been shown by Haisler[34] that for the geometrically nonlinear case, some of these techniques are not applicable because of accuracy or stability considerations. Consequently, this research will only make use of the Euler forward integration method to solve Eq. 10.

Another noteworthy point is that many investigators[7,72,130,10] have concentrated on the method of computing the contribution to the tangent stiffness matrix, $[K^P]$, rather than the more significant problem of which solution procedure should be used. A careful attempt to evaluate the merits of each procedure is not attempted herein. The procedure used in this research is the one popularized by Marcal.[71]

Two other classes of solution procedures which have only recently been formulated[35,76,120] are evaluated in the present study. These solve the equations of the form

$$\{\dot{f}\} + Z\{f\} = 0 \tag{12}$$

and

$$\{\ddot{f}\} + C\{\dot{f}\} + Z\{f\} = 0 \tag{13}$$

where C and Z are scalar quantities. These forms are referred to as <u>self-correcting</u> forms[120] as the unbalance in force returns to zero whereas $\{\dot{f}\} = 0$ tends to drift away from the true solution. For $Z = \frac{1}{\Delta P}$ in Eq. 12 the procedure reduces to the form given in Refs. 43 and 119 which may be interpreted as an incremental form with a one step Newton-Raphson correction.[119] Solutions of the class given by Eq. 13 have not been used for the solution of plasticity problems but have been found to be economical for geometrically nonlinear problems.[35,120]

The literature on combined material and geometric nonlinearities is quite limited and is represented by Refs. 1,7,25,43,68, and 129. Further Ref. 1 reports that some of the matrices are unsymmetric and thus is excluded from consideration. Until very recently the only solution procedures used were of the second class, $\{\dot{f}\} = 0$. It will be shown in the present study that this is one of the most inefficient procedures for the combined problem.

For purposes of discussion the equations of equilibrium for combined material-geometric nonlinearities may be written in the form

$$[K]\{q\} = \{P'\} + \{Q^P\} + \{Q^{NL}\} \tag{14}$$

where, $[K]$, $\{q\}$ and $\{P'\}$ are defined after Eq. 1.

$\{Q^P\}$ pseudo forces due to nonlinear material properties (not the same as $\{Q\}$ in Eq. 1 due to the fact that expressions for the total strain include geometric nonlinearities and plastic effects).

$\{Q^{NL}\}$ = pseudo force due to geometric nonlinearities.

The first derivative of Eq. 14 with respect to the load parameter is written in two separate forms.

$$([K] - [\frac{\partial Q_i^{NL}}{\partial q_j}] - [\frac{\partial Q_i^{P}}{\partial q_j}])\{\dot{q}\} = \{\overline{P}\} \tag{15}$$

and

$$([K] - [\frac{\partial Q_i^{NL}}{\partial q_j}])\{\dot{q}\} = \{\dot{Q}^P\} + \{\overline{P}\} \tag{16}$$

In Refs. 25,43,68 and 129 the solution of Eq. 15 is obtained using a simple forward difference expression. Hofmeister et. al.[43] control the amount of drift by applying a Newton-Raphson iteration (f=0) after a specified number of increments. Felippa[25] formulates the problem in the deformed coordinates of the body. In Refs. 68 the deformed coordinates are referred to but the effects of the deflections and rotations are neglected in the transformation matrix from local to global coordinates. The rigorous and correct derivation of the problem in the Lagrangian coordinates is given by Haisler.[34] In effect the deformed coordinates should not have been referred to in Ref. 68, but this oversight has no bearing on the accuracy of the results.

Armen, Pifko, and Levine[7] solved the combined problem through Eq. 16 where a simple backwards difference expression is used for $\{\dot{Q}^P\}$. The rate of convergence with load increment is much slower for Eq. 16 than for Eq. 15.

Another possible form of $\{\dot{f}\} = 0$, other than that given by Eqs. 15

and 16, is to include the $[\dot{Q}^{NL}]$ term on the right hand side as pseudo forces in the same manner as $[\dot{Q}^{P}]$ is written in Eq. 16. This form is very appealing as it requires only a single inversion of the stiffness matrix. However, a very exhaustive search for a stable numerical procedure presented in Ref. 34 has shown this form to be numerically unstable for significant nonlinearities.

Finally to complete this section, a review of two other aspects of the formulation of the plasticity problem should be reviewed. First, many authors have stated that the initial strain (pseudo plastic forces) may not be used for elastic-perfectly plastic material behavior. Armen, Pifko and Levine[7] have circumvented this difficulty treating the elastic-perfectly plastic case separately.

Actually, when one examines the computational procedure for isotropic hardening it is observed that the same equation may be used with or without strain hardening. Thus the difficulty is not associated with the basic equations but with the fact that rather large increments of strain may be obtained for a small increment of load.[24] These large increments of strain may cause the numerical procedure to obtain stresses and plastic strains which do not lie on the assumed stress strain curve. A computational procedure is presented herein which gives results that lie on the assumed stress strain curve regardless of the strain increment. The improvement in the computational procedure presented herein makes it just as easy to treat the perfectly plastic case as the strain hardening one.

The second aspect is the assumptions which have been made for the variation of the plastic strains over the element. Much of the early research was associated with the solution of plane stress problems using the constant strain triangle. Thus there was no difficulty associated with the evaluation of the pseudo forces and tangent stiffness matrices. They were obtained through a one point numerical integration formula based on the value of the stresses at the centroid of the element. It should be noted that this one point formula does not necessarily imply a constant value of the plastic strain over the element as a linear variation may also be evaluated by a single numerical station.

Felippa[25] used the values of the three corners of a triangular element to evaluate the needed terms in the tangent stiffness approach. For this calculation he used the average value of the stress at the nodes. His example problem showed an appreciable improvement in results for this procedure as compared with the one station numerical integration.

In Ref. 7 a linear variation of the plastic region was assumed for the element. However, these authors assume a linear variation of the plastic strain from the outside surface to zero at some internal station. This assumption is only valid when the entire region lies on the some linear segment of the stress-strain curve.

In Refs. 50 and 51 a linear variation of the matrix relating the plastic strain increment to the total strain increment is used over the meridional length of a shell of revolution element. However, only the displacements of the element under consideration were used to evaluate the strain increments between elements and therefore the plastic strains are not continuous between elements.

Bergan and Clough[10] evaluated the tangent stiffness matrix of a quadrilateral element by first subdividing the element into twelve triangular elements. The values of stresses, etc. at the centroid of each triangular element are assumed to apply over the sub-element. Considerable effort was devoted to developing an efficient algorithm for computing the tangent stiffness matrix. Computer times of 0.54 to 0.69 seconds of CDC 6500 CPU time are quoted for computing the element tangent stiffness matrix using eleven integration stations through the thickness.

Marcal[68] bases the element rigidity matrix for the element on the values of the stresses at the center of the element and then numerically integrates the displacement function to obtain the tangent stiffness matrix. This appears to be a very reasonable approach.

There does not appear to exist any fundamental mathematical proof as to what condition must be satisfied to assure convergence in the computation of plastic strains. However, it seems reasonable to the writers that the stations selected in the numerical integration process should be those where accurate values of the stresses may be computed. If this assertion is true, it implies that only the centroid of the element and the corners where the stresses are computed as average values of surrounding elements may be used in the integration process. The present study evaluates all integrals over the length of the element through strip integration using only values at the mid-length of the element. Convergence studies show that accurate results may be obtained with relatively few elements.

Scope of Research

The purpose of the present study is to evaluate the various solution procedures for material nonlinearities and combined material-geometric nonlinearities. A study has already been conducted for geometrically nonlinear solution procedures. During the course of the present study it was found that many conclusions reached for geometrically nonlinear problems do not apply for material nonlinearities. For example, the incremental approach is quite efficient for plasticity problems but tends to drift appreciably for geometrically nonlinear problems. Furthermore most extrapolation procedures lead to numerical instabilities for geometric problems but are quite stable for plasticity problems. The many differences are discussed in the section on solution procedures.

There is considerable question regarding the correct material flow rule and, in fact, experimental results have shown that some of the flow rules do not adequately describe actual behavior. It is not the purpose of this research to dwell upon the development of material laws or even to assess which is the most correct. Rather, this paper concerns itself primarily with the development of efficient solution techniques for solving the governing equilibrium equation. The Von Mises yield criterion with isotropic hardening and elastic unloading is assumed in this research. However, it is believed that the conclusions reached in this report regarding the efficiency of computational procedures will be valid regardless of the yield condition and flow rule used.

FORMULATION OF EQUILIBRIUM EQUATIONS

The purpose of this section is to present the formulation of the equations of equilibrium as related to the various types of solution procedures. The formulation as presented herein is symbolic and consequently applicable to any type of incremental plasticity law and any of the numerous finite element models. The plasticity law for isotropic hardening of shells of revolution under axisymmetric loads is presented in the next section and the details of the computational procedure for shells of revolution are presented in the following section.

Starting with the equilibrium equations for large deflections and large strains in terms of the undeformed coordinates, then multiplying by virtual displacements in the x, y, and z directions and applying the divergence theorem yields:[34]

$$\int_V \lfloor \sigma^* \rfloor \{\delta\varepsilon\} dV = \delta W^* \tag{17}$$

where $\lfloor \sigma^* \rfloor$ = Stress tensor referred to the undeformed areas

$\{\delta\varepsilon\}$ = Virtual change in the total strain tensor including the contribution due to large deflections.

δW^* = Virtual work of external and body forces as computed in the deformed coordinate system.

V = Volume of undeformed body.

Equation 17 is valid for any type of material and for large deflections and large strains. The present investigation is restricted to small strains and thus

$$\lfloor \sigma^* \rfloor = \lfloor \sigma \rfloor \tag{18}$$

where $\lfloor \sigma \rfloor$ is the usual engineering definition of stress and is related to the elastic strains through a matrix [D].

$$\{\sigma\} = [D]\{\varepsilon^e\} \tag{19}$$

As small strains are assumed the total strain is the linear superposition of the various components.

$$\{\varepsilon\} = \{\varepsilon^e\} + \{\varepsilon^P\} + \{\varepsilon^T\} + \ldots \tag{20}$$

where $\{\varepsilon\}$ = total strain

$\{\varepsilon^e\}$ = elastic strain

$\{\varepsilon^P\}$ = plastic strain

$\{\varepsilon^T\}$ = thermal strain

Solving Eq. 20 for the elastic strain, substituting into Eq. 19 and substituting the result into Eq. 17 yields:

$$\int_V (\lfloor \varepsilon \rfloor - \lfloor \varepsilon^P \rfloor - \lfloor \varepsilon^T \rfloor + \ldots)[D]\{\delta\varepsilon\}dV = \delta W^* \tag{21}$$

For some problems the potential due to external forces may be a higher order function of the displacements; but, as usual, the assumption of a first order function of the displacements is assumed herein.

$$W^* = \lfloor q \rfloor \{P'\} \tag{22}$$

Thus,

$$\delta W^* = \lfloor P' \rfloor \{\delta q\} \tag{23}$$

Taking the variation with respect to generalized coordinate q_i

yields the equation of equilibrium:

$$\int \lfloor \frac{\partial \varepsilon}{\partial q_i} \rfloor [D] \{\varepsilon\} dV - \int \lfloor \frac{\partial \varepsilon}{\partial q_i} \rfloor [D]\{\varepsilon^P\}dV - \int \lfloor \frac{\partial \varepsilon}{\partial q_i} \rfloor [D]\{\varepsilon^T\}dV - \ldots = P_i' \tag{24}$$

where P_i' is the generalized force associated with the external applied loads.

It is convenient to write the total strain as

$$\varepsilon = \varepsilon_L + \varepsilon_{NL} \tag{25}$$

where ε_L and ε_{NL} are the linear and nonlinear contributions, respectively, to the total strain. Substituting Eq. 25 into Eq. 24 and expanding the first term on the left hand side, yields

$$\int \lfloor \frac{\partial \varepsilon_L}{\partial q_i} \rfloor [D] \{\varepsilon_L\} dV + \int \lfloor \frac{\partial \varepsilon}{\partial q_i} \rfloor [D] \{\varepsilon_{NL}\}dV + \int \lfloor \frac{\partial \varepsilon_{NL}}{\partial q_i} \rfloor [D] \{\varepsilon_L\} dV$$

$$- \int \lfloor \frac{\partial \varepsilon}{\partial q_i} \rfloor [D]\{\varepsilon^P\}dV - \int \lfloor \frac{\partial \varepsilon}{\partial q_i} \rfloor [D] \{\varepsilon^T\}dV - \ldots = P_i' \tag{26}$$

The first term in Eq. 26 gives the contribution to the usual linear stiffness matrix, K, times the generalized coordinates, q_i. The remaining terms may be combined to yield

$$\Sigma_j k_{ij} \; q_j = P_i' - \int \lfloor \frac{\partial \varepsilon_{NL}}{\partial q_i} \rfloor [D] \{\varepsilon_L\}dV - \int \lfloor \frac{\partial \varepsilon}{\partial q_i} \rfloor [D]\{\varepsilon_{NL} - \varepsilon^P - \varepsilon^T \; \ldots\}dV \tag{27}$$

Writing Eq. 27 for each and every degree of freedom i yields the complete set of equilibrium equations:

$$[K]\ \{q\} = \{P'\} + \{Q^*\} \tag{28}$$

where

$$Q_i^* = -\int \left\lfloor \frac{\partial \varepsilon_{NL}}{\partial q_i} \right\rfloor [D]\ \{\varepsilon_L\}dV - \int \left\lfloor \frac{\partial \varepsilon}{\partial q_i} \right\rfloor [D]\{\varepsilon_{NL} - \varepsilon^P - \varepsilon^T \quad - \ . \ . \ . \}dV \tag{29}$$

The last term on the right side of Eq. 28 is generally called the pseudo force and is a function of the unknown displacements.

In Eq. 29 the pseudo forces due to material and geometric nonlinearities are included together instead of separating them into components as indicated in Eq. 14. The separate terms are given as:

$$Q_i^P = \int \left\lfloor \frac{\partial \varepsilon}{\partial q_i} \right\rfloor [D]\{\varepsilon^P\}dV \tag{30}$$

$$Q_i^{NL} = -\int \left\lfloor \frac{\partial \varepsilon_{NL}}{\partial q_i} \right\rfloor [D]\{\varepsilon_L\}dV - \int \left\lfloor \frac{\partial \varepsilon}{\partial q_i} \right\rfloor [D]\{\varepsilon_{NL}\}dV = -\int \left\lfloor \frac{\partial \varepsilon_L}{\partial q_i} \right\rfloor [D]\{\varepsilon_{NL}\}dV$$
$$- \int \left\lfloor \frac{\partial \varepsilon_{NL}}{\partial q_i} \right\rfloor [D]\{\varepsilon\}dV \tag{31}$$

The last form of Eq. 31 is the more efficient from the computational point of view when only geometric nonlinearities are considered. Further for a unstiffened shell the pseudo forces due to geometric nonlinearities may be integrated exactly through the thickness provided nonlinearities in the curvature expression are excluded. Thus, the assumption is made that

$$\{\varepsilon\} = \{e\} + \{\varepsilon_{NL}\} + z\{\chi\} \tag{32}$$

where $\{e\}$ are the usual expression for the linear membrane strains,

$\{\chi\}$ are the changes in curvature, and z is the distance from the mid-surface of the shell. In light of Eq. 25, it should be noted that the first and third terms of Eq. 32 constitute $\{\varepsilon_L\}$, i.e., $\{\varepsilon_L\} = \{e\} + z\{\chi\}$. Substituting Eq. 32 into Eq. 31 and integrating through the thickness yields

$$Q_i^{NL} = -\ t \int \lfloor \frac{\partial e}{\partial q_i} \rfloor \ [D]\{\varepsilon_{NL}\}\ dA - t \int \ \lfloor \frac{\partial \varepsilon_{NL}}{\partial q_i} \rfloor \ [D](\{e\} + \{\varepsilon_{NL}\})dA \qquad (33)$$

where t is the thickness and dA is a differential of surface area.

Several points should be noted pertaining to Eqs. 29-33. First the pseudo forces due to initial and thermal strains are not constants as the term $\lfloor \frac{\partial \varepsilon}{\partial q} \rfloor$ depends on the displacements($\{\varepsilon_{NL}\}$ is a second order function of the displacements). Second, if the midsurface of the shell yields it is a simple matter to include part of the contributions due to geometric nonlinearities as given by the second term of Eq. 29 and the rest by the last term in Eq. 33 with the term $\{\varepsilon_{NL}\}$ being omitted. This has been found to save appreciably on computer run times.

At this point, attention must be focused on evaluating the integrals in Eq. 29 . There has recently been considerable disagreement over this point between finite element and finite difference researchers. Originally, finite difference equations were formulated from governing differential equations. Recently, several investigators (Ref. 55) have turned toward finite differencing of the strain energy. Most finite element formulations have tended toward exact, closed form integration of the strain energy; particularly in obtaining the stiffness matrix. The present research seeks to combine and utilize the best aspects of both approaches.

In the present formulation, the linear stiffness matrix is evaluated exactly in closed form. This is based on the contention that the element stiffness matrix should be represented as accurately as possible. However, in treating the nonlinearities, certain simplifying approximations are used which greatly reduce computation time. First, the strains, rotations, and change in curvatures are evaluated using finite difference expressions and are written in terms of the global generalized coordinates. The partial derivatives of the strains with respect to the generalized coordinates are then easily evaluated. Simple strip integration is used over the area of the element while more exact Euler or Simpson integration is used through the thickness. With the present formulation, the plastic strain effects, thermal effects, and so on are taken into account by simply computing and subtracting the appropriate component in the last term of Eq. 29 before performing the integration.

The key to the computational procedure consists of evaluating the partial derivatives of the strains, rotations, and changes in curvature with respect to the generalized coordinates. These partial derivatives are used quite frequently throughout the entire analysis. For example, the strains, rotations and curvatures are simply linear functions of the generalized coordinates and are obtained by summing the derivatives times the respective generalized coordinates. The details of the computational procedure for shell analysis will be presented in a later section.

In some solution procedures such as the Newton-Raphson method or the first order differential equation approach it is necessary to compute a contribution to the tangent stiffness matrix. From Eq. 28, letting {P'} = P{$\bar{P}$} and taking the derivative of Eq. 28 with respect to the load parameter P, yields.

$$[K]\{\dot{q}\} = \{\bar{P}\} + \{\dot{Q}^*\} \tag{34}$$

where the dot denotes differentiation with respect to P. Further, using chain rule differentiation on {$\dot{Q}^*$}, Eq. 34 may be written as

$$([K] + [K^*])\ \{\dot{q}\} = \{\bar{P}\} \tag{35}$$

where

$$K^*_{ij} = -\frac{\partial Q^*_i}{\partial q_j} \tag{36}$$

Although the development of the [K*] matrix has been shown only for the above first-order differential equation form, this matrix is necessary in all solution procedures involving the use of a tangent stiffness matrix including the Newton-Raphson method and others.

Applying Eq. 36 to Eq. 29 yields the terms of the contribution to the tangent stiffness matrix as

$$\begin{aligned} K^*_{ij} = -\frac{\partial Q^*_i}{\partial q_j} &= \int \lfloor \frac{\partial^2 \varepsilon_{NL}}{\partial q_i \partial q_j} \rfloor\ [D]\ \{\varepsilon_L\}\ dV \\ &+ \int \lfloor \frac{\partial \varepsilon_{NL}}{\partial q_i} \rfloor\ [D]\ \{\frac{\partial \varepsilon_L}{\partial q_j}\}\ dV + \int \lfloor \frac{\partial^2 \varepsilon_{NL}}{\partial q_i \partial q_j} \rfloor\ [D]\{\varepsilon_{NL} - \varepsilon^P - \varepsilon^T \quad - \ . \ . \ .\}dV \\ &+ \int \lfloor \frac{\partial \varepsilon}{\partial q_i} \rfloor\ [D]\ \{\frac{\partial \varepsilon_{NL}}{\partial q_j}\}\ dV \\ &- \int \lfloor \frac{\partial \varepsilon}{\partial q_i} \rfloor\ [D]\ \{\frac{\partial \varepsilon^P}{\partial q_j}\}\ dV \end{aligned} \tag{37}$$

It should be pointed out that the thermal and initial strains are assumed to be independent of the displacements. Further, in writing the third term on the right hand side of Eq. 37, the fact that the second partial of the total strain yields only that contribution due to the nonlinear terms has been accounted for, that is,

$$\frac{\partial^2 \varepsilon}{\partial q_i \partial q_j} = \frac{\partial^2 \varepsilon_{NL}}{\partial q_i \partial q_j} \tag{38}$$

It should be noted in Eq. 37 that there are contributions due to thermal, initial, etc. strains. However, the first and third terms in Eq. 37 may be combined to yield:

$$\begin{aligned} &\int \left\lfloor \frac{\partial^2 \varepsilon_{NL}}{\partial q_i \partial q_j} \right\rfloor [D] \{\varepsilon_L\} \, dV \\ &+ \int \left\lfloor \frac{\partial^2 \varepsilon_{NL}}{\partial q_i \partial q_j} \right\rfloor [D] \{\varepsilon_{NL} - \varepsilon^P - \varepsilon^T \quad - \ . \ . \ .\} \, dV \\ &= \int \left\lfloor \frac{\partial^2 \varepsilon_{NL}}{\partial q_i \partial q_j} \right\rfloor [D] \{\varepsilon^e\} \, dV \end{aligned} \tag{39}$$

where $\{\varepsilon^e\}$ is the elastic component of the strain defined in Eq. 20. For those readers accustomed to evaluating the tangent stiffness matrix based on the concept of initial stresses it may be noted that $[D]\{\varepsilon^e\}$ are simply the stresses that exist in the structure.

Regardless of the incremental or plasticity theory being used it is always possible to write a relationship between the increment of plastic strain and the increment of total strain

$$\{d\varepsilon^P\} = [C]\ \{d\varepsilon\} \tag{40}$$

where the matrix [C] depends upon the previous state of stress and strain history. Dividing each side of Eq. 40 by dq_j yields the desired relation

$$\{\frac{\partial\varepsilon^P}{\partial q_j}\} = [C]\ \{\frac{\partial\varepsilon}{\partial q_j}\} \tag{41}$$

Using the relations given by Eqs. 39 and 41 in Eq. 37 and making use of Eq. 25 in the next to the last term of Eq. 37 yields the final somewhat simplified form for K^*_{ij}

$$\begin{aligned} K^*_{ij} = &\int \left(\lfloor\frac{\partial\varepsilon_{NL}}{\partial q_i}\rfloor\ [D]\ \{\frac{\partial\varepsilon_L}{\partial q_j}\} + \lfloor\frac{\partial\varepsilon_L}{\partial q_i}\rfloor\ [D]\ \{\frac{\partial\varepsilon_{NL}}{\partial q_j}\}\right) dV \\ &+ \int \lfloor\frac{\partial\varepsilon_{NL}}{\partial q_i}\rfloor\ [D]\ \{\frac{\partial\varepsilon_{NL}}{\partial q_j}\}\ dV + \int \lfloor\frac{\partial^2\varepsilon_{NL}}{\partial q_i \partial q_j}\rfloor\ [D]\{\varepsilon^e\}\ dV \\ &- \int \lfloor\frac{\partial\varepsilon}{\partial q_i}\rfloor\ [D]\ [C]\ \{\frac{\partial\varepsilon}{\partial q_j}\}\ dV \end{aligned} \tag{42}$$

It is noted in Eq. 42 that the matrix [K*] is symmetric provided that the product [D] [C] is symmetric. This is true for the case of isotropic hardening used herein. Further, it is noted that many of the same derivatives appear in Eq. 42 that appeared in Eqs. (29-33). Thus, in reality, the computation of the matrix [K*] is a minor addition to the computational procedure needed for the pseudo forces.

Equation 42 is one form that may be used to compute the contribution to the tangent stiffness matrix. It may be further simplified by integrating the first two integral terms through the thickness; but, the last two terms must be integrated numerically through the thickness. It is believed that this form is the most efficient one for computational procedures since $[D]\{\varepsilon^e\}$ is equal to the stresses which must be computed in conjunction with the evaluation or checking for plastic strains. However, it has a disadvantage in that the third integral is in reality due to a combination of geometric and material nonlinearities. For that reason, the present study separates the two effects and writes the third term as

$$\int \lfloor \frac{\partial^2 \varepsilon_{NL}}{\partial q_i \partial q_j} \rfloor [D]\{\varepsilon^e\}\, dV = \int \lfloor \frac{\partial^2 \varepsilon_{NL}}{\partial q_i \partial q_j} \rfloor [D](\{\varepsilon\} - \{\varepsilon^P\})\, dV \qquad (43)$$

where no initial, thermal, creep, etc. strains are considered for the present investigation in writing Eq. 43. Through the use of Eq. 43, Eq. 42 for K^*_{ij} may be written as

$$K^*_{ij} = K^{NL}_{ij} + K^P_{ij} \qquad (44)$$

where

$$K^{NL}_{ij} = \int \left(\lfloor \frac{\partial \varepsilon_{NL}}{\partial q_i} \rfloor [D] \{\frac{\partial \varepsilon_L}{\partial q_j}\} + \lfloor \frac{\partial \varepsilon_L}{\partial q_i} \rfloor [D] \{\frac{\partial \varepsilon_{NL}}{\partial q_j}\} \right) dV$$

$$+ \int \lfloor \frac{\partial \varepsilon_{NL}}{\partial q_i} \rfloor [D] \{\frac{\partial \varepsilon_{NL}}{\partial q_j}\}\, dV + \int \lfloor \frac{\partial^2 \varepsilon_{NL}}{\partial q_i \partial q_j} \rfloor [D]\{\varepsilon\}\, dV \qquad (45)$$

$$K^P_{ij} = -\int \lfloor \frac{\partial^2 \varepsilon_{NL}}{\partial q_i \partial q_j} \rfloor [D]\{\varepsilon^P\} dV - \int \lfloor \frac{\partial \varepsilon}{\partial q_i} \rfloor [D][C] \{\frac{\partial \varepsilon}{\partial q_j}\}\, dV \qquad (46)$$

Equation 45 may be further simplified by using Eq. 32 and integrating through the thickness of the shell

$$K_{ij}^{NL} = t \int \left(\lfloor \frac{\partial \varepsilon_{NL}}{\partial q_i} \rfloor [D] \{ \frac{\partial e}{\partial q_j} \} + \lfloor \frac{\partial e}{\partial q_i} \rfloor [D] \{ \frac{\partial \varepsilon_{NL}}{\partial q_j} \} \right) dA$$

$$+ t \int \lfloor \frac{\partial \varepsilon_{NL}}{\partial q_i} \rfloor [D] \{ \frac{\partial \varepsilon_{NL}}{\partial q_j} \} dA$$

$$+ t \int \lfloor \frac{\partial^2 \varepsilon_{NL}}{\partial q_i \partial q_j} \rfloor [D] (\{e\} + \{\varepsilon_{NL}\}) dA \qquad (47)$$

It should be pointed out that the form for the pseudo forces represented by Eqs. 29-31 and for contributions to the tangent stiffness matrices represented by Eqs. 42, 45, 46, or 47 may be evaluated using a wide variety of numerical integration procedures. For example, the general form for the pseudo forces due to material nonlinearities given by Eq. 30 may be represented as

$$Q_i^P = \Delta z \ A \ \sum_{\text{area}} W_m^A \sum_{\text{thickness}} W_n^z \lfloor \frac{\partial \varepsilon}{\partial q_i} \rfloor [D] \{\varepsilon^P\} \qquad (48)$$

where

Q_i^P = element plastic contribution (note that no distinction is made between element contribution and total)

Δz = increment through thickness

A = Area of element

W^A, W^z = weighting function which depends on the type of numerical integration being used.

Finally, it is a simple matter to show that for material nonlinearities only, the present formulation for the tangent stiffness matrix reduces to that given by Marcal.[71] For linear strain-displacement relations, the strains may be written as

$$\{\varepsilon\} = [B]\ \{q\} \tag{49}$$

where {q} are the nodal displacements. Substituting Eq. 49 into Eq. 46 and noting that the first term of Eq. 46 is identically zero yields

$$[K^P] = - \int [B]^T\ [D][C][B]\ dV \tag{50}$$

When this term is added to the usual expression for the element stiffness matrix, the complete tangent stiffness matrix may be written as

$$[K] + [K^P] = \int [B]^T\ ([D] - [D][C])\ [B]\ dV \tag{51}$$

This is precisely the same form given by Marcal in a slightly different notation.

PLASTICITY RELATIONS

The formulation which has been presented in the previous section is valid for any incremental law of plasticity. All that is needed is a method for the computation of the total plastic strain and a relationship between the increments of plastic strain and the total strain increment. The purpose of this section is not to present any new information regarding plasticity relations but is simply to present the formulation and a discussion of the Von Mises yield condition and associated isotropic hardening law used in the present investigation. More detailed survey papers have been referred to in the introduction. This section first presents a rather intuitive approach to isotropic hardening as presented by Crandall and Dahl[15] followed by the mathematical formulation which has been popularized by Marcal.[71]

For an acceptable plasticity rule three separate conditions must be specified. These conditions are the stress state at which yielding occurs, some rule that allows the calculation of the plastic strains or strain increment, and a hardening rule.

It is known from dislocation theory that plastic strains are a result of the movement of dislocations. These movements are caused by shear stresses and are for all practical purposes independent of the mean normal stress. The yield condition which represents this phenomena is the Tresca yield condition as shown in Fig. 1, along with the Von Mises yield condition.[15] In this figure, σ_s and σ_θ are the principal stresses for a plane stress condition. It is noted that the experimental data is in better agreement with the smooth Von Mises yield condition than the Tresca yield condition. For this reason, and ease of handling, most engineers prefer the Von Mises yield condition for metals and this condition is used in the present research. The Von Mises yield condition is when the shear stress on an octahedral plane or the energy of distortion exceeds prescribed values.[15,39] The equation representing this condition for a state of plane stress for an isotropic material is given by:

$$\phi = \sigma_s^2 + \sigma_\theta^2 - \sigma_s \sigma_\theta + 3 \sigma_{s\theta}^2 - \bar{\sigma}^2 \qquad (52)$$

where σ_s, σ_θ, $\sigma_{s\theta}$ are the normal stresses in the s and θ directions and the shear stress respectively.

$\bar{\sigma}$ is the yield stress obtained for a uniaxial test specimen and may increase due to the hardening of the material.

Yielding occurs when $\phi \geq 0$ with an elastic state existing for all values of $\phi < 0$.

Considering the fact that the yield surface is expanding ($\bar{\sigma}$ increasing) it is possible to represent all subsequent yielding by $\phi = 0$ and consequently it becomes possible to solve for $\bar{\sigma}$.

$$\bar{\sigma} = (\sigma_s^2 + \sigma_\theta^2 - \sigma_s \sigma_\theta + 3 \sigma_{s\theta}^2)^{1/2} \qquad (53)$$

The flow rule relating the increment of plastic strain to the state of stress for isotropic hardening is of the same form as the expression for elastic strains except Poisson's ratio is equal to 1/2 indicating an incompressible body during plastic deformations. For the plane stress condition the increments of plastic strains are given by:[15]

$$d\varepsilon_s^P = \frac{d\bar{\varepsilon}^P}{\bar{\sigma}} [\sigma_s - .5\ \sigma_\theta]$$

$$d\bar{\varepsilon}_\theta^P = \frac{d\bar{\varepsilon}^P}{\bar{\sigma}} [\sigma_\theta - .5\ \sigma_s] \tag{54}$$

$$d\varepsilon_{s\theta}^P = 3\ \frac{d\bar{\varepsilon}^P}{\bar{\sigma}}\ \sigma_{s\theta}$$

where $d\varepsilon_s^P$, $d\varepsilon_\theta^P$, and $d\varepsilon_{s\theta}^P$ are the increments of plastic strain and $d\bar{\varepsilon}^P$ = increment of equivalent uniaxial plastic strain. It is noted that the term $\frac{d\bar{\varepsilon}^P}{\bar{\sigma}}$ is the proportionality factor for a plastic strain increment as 1/E is the factor for elastic strains.

This rather intuitive presentation for the plastic strain increment has a very sound mathematical foundation which is commonly referred to as the normality condition. The condition simply states that the direction of the plastic strain increment must be normal to the ϕ = constant surface. Thus the plastic strain increments are given the compact notation as

$$\{d\varepsilon^P\} = d\bar{\varepsilon}^P \left\{\frac{\partial\bar{\sigma}}{\partial\sigma}\right\} \tag{55}$$

Finally the hardening rule for the material is simply the relation between the plastic strain increment $d\bar{\varepsilon}^P$ and stress increment $d\bar{\sigma}$ for the uniaxial case.

$$d\bar{\sigma} = H'\ d\bar{\varepsilon}^P \tag{56}$$

For the case of piecewise linear strain hardening shown in Fig. 2, it is easy to show that:

$$H' = \frac{E\,E_T}{E-E_T} \tag{57}$$

The other relation needed to complete the formulation is the equation relating the increment of stress to the increment of plastic strain.

$$\{d\sigma\} = [D]\{d\varepsilon^e\} = [D](\{d\varepsilon\} - \{d\varepsilon^P\}) \tag{58}$$

Premultiplying Eq. 58 by $\lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\rfloor$ and using Eq. 55 and 56 yields:

$$\lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\rfloor\ \{d\sigma\} = d\bar{\sigma} = H'd\bar{\varepsilon}^P = \lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\rfloor\ [D](\{d\varepsilon\} - d\bar{\varepsilon}^P\ \{\frac{\partial\bar{\sigma}}{\partial\sigma}\}) \tag{59}$$

Solving Eq. 59 for $d\bar{\varepsilon}^P$ yields

$$d\bar{\varepsilon}^P = \frac{\lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\rfloor[D]\{d\varepsilon\}}{H' + \lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\rfloor\ [D]\{\frac{\partial\bar{\sigma}}{\partial\sigma}\}} \tag{60}$$

After computing the uniaxial plastic strain increment, the column vector of plastic strain increments is computed through Eq. 55 and then the stress increment is computed from Eq. 58. This stress increment is added to the stresses existing at the beginning of the increment and the resulting values are used to compute the matrix [C] (Eq. 40 relating the increments of plastic strains to the increment of total strain). Substituting Eq. 60 into Eq. 55 yields:

$$\{d\varepsilon^P\} = \frac{\{\frac{\partial\bar{\sigma}}{\partial\sigma}\}\ \lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\rfloor[D]\{d\varepsilon\}}{H' + \lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\rfloor[D]\{\frac{\partial\bar{\sigma}}{\partial\sigma}\}} \tag{61}$$

thus

$$[C] = \frac{\{\frac{\partial \bar{\sigma}}{\partial \sigma}\} \lfloor \frac{\partial \bar{\sigma}}{\partial \sigma} \rfloor [D]}{H' + \lfloor \frac{\partial \bar{\sigma}}{\partial \sigma} \rfloor [D] \{\frac{\partial \bar{\sigma}}{\partial \sigma}\}} \tag{62}$$

Several points should be emphasized in the computational procedure. First, as presented above, the matrix [C] should be based on the stresses at the end of the known increment. Otherwise the plasticity matrix will lag by one increment of stress and consequently will impede convergence.

The second point is that an examination of the derivation procedure reveals that every step is valid for a perfectly plastic material ($H' = 0$) as well as a strain hardening one ($H' \neq 0$). However, it was found that for small values of H' or large load increments, the computational procedure yields uniaxial stress-plastic strains which deviates appreciably from the assumed stress-strain diagram. This difficulty is avoided herein by dividing the total strain increment into a number of smaller equal strain increments. The same computational procedure outlined above is used for the small increments with stresses being updated after each cycle of computation. This is described in detail in the next section.

Unloading occurs when the value of $d\bar{\varepsilon}^P$ computed in Eq. 60 is less than zero. When this occurs $d\bar{\varepsilon}^P$ and all terms in the matrix [C] are set equal to zero. Furthermore no additional plastic straining is permitted until $\bar{\sigma}$ exceeds its previously maximum value. It should be noted that this gives a negative Bauschinger effect which is unfortunately the case for isotropic hardening.

The treatment of the transition region between elastic and plastic behavior is the same one presented by Krieg and Duffey[54] and Yamada[131]. It is presented here for case of axisymmetric deflections of shells of revolution ($\sigma_{s\theta} = 0$).

Let σ_s and σ_θ be the values of the stresses at the previous load where the point under consideration is purely elastic and $\Delta\sigma_s^e$, $\Delta\sigma_\theta^e$ be the increments of stress based on elastic material but which cause yielding of the point under consideration.

$$\phi(\sigma + \Delta\sigma^e) > 0 \tag{63}$$

Further, let k be a constant for which the stress increments $k\ \Delta\sigma_s^e$ and $k\ \Delta\sigma_\theta^e$ give $\phi = 0$, i.e. on the yield surface. Then

$$(\sigma_s + k\ \Delta\sigma_s^e)^2 + (\sigma_\theta + k\ \Delta\sigma_\theta^e)^2 - (\sigma_s + k\ \Delta\sigma_s^e)\ (\sigma_\theta + k\ \Delta\sigma_\theta^e) - \sigma_y^2 = 0 \tag{64}$$

where σ_y is the yield stress. Solving Eq. 64 for the constant k yields

$$k = \frac{-B + \sqrt{B^2 - 4\ \phi_0\ \Delta\ \phi^e}}{2\Delta\phi^e} \tag{65}$$

where

$$\phi_0 = \sigma_s^2 + \sigma_\theta^2 - \sigma_s\ \sigma_\theta - \sigma_y^2$$

$$B = \Delta\ \sigma_s^e\ (2\sigma_s - \sigma_\theta) + \Delta\sigma_\theta^e\ (2\sigma_\theta - \sigma_s)$$

$$\Delta\phi^e = (\Delta\sigma_s^e)^2 + (\Delta\sigma_\theta^e)^2 - \Delta\sigma_s^e\ \Delta\sigma_\theta^e \tag{66}$$

After solving for k the values of the stress are advanced to those at which yielding first occurs and the total strain increment is reduced by (1. -k). The computations then proceed from Eq. 60 through 62 as presented before. Subsequent changes in the values of H' are controlled by changing H' whenever $\bar{\varepsilon}^p$ exceeds prescribed values. As the strain increment is always required to be small and drastic changes do not occur after initial yielding this procedure has been found to be completely satisfactory.

COMPUTATIONAL PROCEDURE

As stated in a previous section the basic philosophy used herein is to obtain a highly refined stiffness matrix based on linear theory and to approximate the nonlinear terms, both geometric and material, so as to produce an efficient computational procedure. The purpose of this section is to present the details of the computational procedure for evaluating the pseudo forces and contributions to the tangent stiffness matrix. For simplicity attention is restricted to axisymmetric deflections of shells of revolution; but the procedure has been developed and is operational for the asymmetric deflections of shells of revolution. This extension simply requires the inclusion of a shear stress and numerical integration around the circumference of the shell.

The element stiffness matrix is evaluated by using cubic functions in the meridional distance, s, for all displacements and eliminating the two (four for asymmetric case) additional degrees of freedom through static condensation. Mebane and Stricklin[78] have shown that this yields an element stiffness matrix with excellent rigid body characteristics even for relatively large subtended angles. The curvature of the shell in the meridional direction is represented in the manner given by Stricklin, et. al.[116] The net result of these refinements is a highly accurate element stiffness matrix which has been shown to yield accurate results with only a few elements through numerous solutions of both static and dynamic problems over the last three years. The refinements mentioned above do require additional computer time but the total computer time required to generate the global stiffness matrix is rather trivial when compared with the computer time required to solve nonlinear problems.

The best way to describe the representation of the nonlinearities is to simply state that finite difference expressions are used with the various terms being evaluated at the middle of the element in the meridional direction. This finite difference representation does not require points external to the shell and discontinuities in slopes between elements may exist. This representation was first presented by Stricklin, Haisler, et. al.[118] in 1968 and has undergone several degrees of refinement since that time. At present, simple finite difference expressions are used in conjunction with conical frustum elements. The conical frustum elements are used to circumvent the rigid body displacement problems which are associated with curved elements and lower order displacement functions.

A critical problem when material nonlinearities are considered is the storage requirements. In the computational procedure presented here only the values of the plastic strains, equivalent uniaxial plastic strain, and the maximum value of the uniaxial stress are required for each element and each numerical integration station through the thickness. The stresses are computed based on the value of the total strain and the values of the plastic strains.

When the midsurface of an element has yielded it is possible to include part of the geometric nonlinearities with the material nonlinearities and thus speed up the computational procedure. However the procedure described here treats the material and geometric contributions separately.

The coordinate system and displacements for a shell of revolution element are shown in Fig. 3. S and θ are the meridional distance and circumferential angle respectively. ϕ is the slope measured from the axis of the shell. u and w are the displacements in the meridional and normal directions. The generalized coordinates, $q_1 \ldots q_8$, are measured in cylindrical coordinates. The displacements in the circumferential direction in Fig. 3 (q_2 and q_6) are suppressed in the present analysis.

The strains at any point through the thickness of the element are given by:

$$\varepsilon_s = e_s + \frac{1}{2} e_{13}^2 + z\,\chi_s$$

$$\varepsilon_\theta = e_\theta + z\,\chi_\theta \tag{67}$$

where z = outward distance from neutral surface. It is noted that only geometric nonlinearities due to a rotation e_{13} are considered. The linear expressions for the strains, rotations, and curvatures for conical frustum elements are given as:

$$e_s = \frac{du}{ds}$$

$$e_\theta = \frac{1}{r}(u \sin\phi + w \cos\phi)$$

$$e_{13} = \frac{dw}{ds} \tag{68}$$

$$\chi_s = -\frac{de_{13}}{ds}$$

$$\chi_\theta = -\frac{\sin\phi}{r} e_{13}$$

The various terms in Eqs. 68 are shown in Fig. 3.

The equations relating the stress to the elastic strains are given by:

$$\begin{Bmatrix} \sigma_s \\ \\ \sigma_\theta \end{Bmatrix} = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu \\ \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_s^e \\ \\ \varepsilon_\theta^e \end{Bmatrix} \tag{69}$$

Thus the matrix [D] in Eq. 19 is

$$[D] = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \tag{70}$$

Writing Eqs. 68 in finite difference form is a simple matter with the exception of the term e_{13}. This term has two possible representations which are in terms of w at the ends of the element and the average values of the rotations, q_4 and q_8, at the ends of the elements. The first of these representations is the proper one as a finite difference expression for a first derivative may be obtained by assuming a linear displacement function. The representation of e_{13} in terms of q_4 and q_8 was tried some time ago and found to yield erroneous results. The finite difference expressions used in this research are:

$$e_s = \frac{-q_1 \cos\phi - q_3 \sin\phi + q_5 \cos\phi + q_7 \sin\phi}{L}$$

$$e_\theta = \frac{q_3 + q_7}{2\,r}$$

$$e_{13} = \frac{q_1 \sin\phi - q_3 \cos\phi - q_5 \sin\phi + q_7 \cos\phi}{L} \tag{71}$$

$$\chi_s = \frac{q_4 - q_8}{L}$$

$$\chi_\theta = -\frac{\sin\phi}{r}\, e_{13}$$

where ϕ = slope of conical frustum element

r = radial coordinate to middle of element

L = length of element

Equations 67-71 are the basic expressions needed to compute the pseudo forces due to material (Eq. 30) and geometric (Eq. 33) nonlinearities and the contributions to the tangent stiffness matrix (Eqs. 46 and 47). At zero load, the displacements are zero and after the first increment of load the displacements or the first guess for the displacements are taken as the linear solution. Thus, there are two known values for the displacement after any load increment. The computational procedure is the same for all elements. Therefore the following description is for any element.

A. From Eqs. 71 compute the partial derivatives of e_s, e_θ, e_{13}, χ_s, and χ_θ with respect to each generalized coordinate.

Example:

$$\frac{\partial e_s}{\partial q_1} = -\frac{\cos\phi}{L} \tag{72}$$

<u>The computation of these partial derivatives is the key to an efficient computational procedure</u> as they are used frequently in the computations.

B. Using the values of the partial derivatives in step A and the latest known values for the generalized displacements, compute the values of e_s, e_θ, χ_s, and χ_θ.

Example:

$$\chi_s = \frac{\partial \chi_s}{\partial q_4} q_4 + \frac{\partial \chi_s}{\partial q_8} q_8 \tag{73}$$

C. Compute the terms of the matrix [D] given by Eq. 70.

D. Check to see if the element has previously yielded by determining if equivalent uniaxial plastic strain at the upper or lower surface is greater than zero. If previous yielding has occurred, skip step E.

E. Check for yielding at upper and lower surface based on elastic behavior. This consists of evaluating Eq. 67, substituting the result into Eq. 69, and checking the yield criteria given by Eq. 52 with $\sigma_{s\theta} = 0$. If yielding does not occur, skip steps F and G.

F. Compute increment of strains, curvatures, and rotations based on displacements at current and previous load step. For example,

$$\Delta e_s = \frac{\partial e_s}{\partial q_1} \Delta q_1 + \frac{\partial e_s}{\partial q_3} \Delta q_3 + \frac{\partial e_s}{\partial q_5} \Delta q_5 + \frac{\partial e_s}{\partial q_7} \Delta q_7 \tag{74}$$

G. For each numerical integration station through the thickness perform the following calculations:

1. If yielding has not occurred at previous load increment check for yielding. If yielding does not occur advance to next numerical integration station and repeat the present step.
2. If this is the first time yielding occurs at the numerical integration station compute the factor k as given in the section on plasticity relations. Advance stresses to yield stress and reduce strain increment by (1.0 - k).

3. If element has yielded at the previous load increment, compute the stresses at the beginning of the present load increment. This is accomplished by subtracting the increments of strains, rotations, and curvatures computed in step F from the total strains computed in step B and subtracting the previous values of the plastic strains to yield the elastic strains. The stresses are then computed using the matrix [D] and the elastic strains.

4. Compute increment of equivalent total strain as given by:

$$\Delta\bar{\varepsilon} = \frac{2}{\sqrt{3}} [(\Delta\varepsilon_s)^2 + (\Delta\varepsilon_\theta)^2 + \Delta\varepsilon_s\Delta\varepsilon_\theta]^{1/2} \tag{75}$$

It is noted that the equivalent strain increment is based on the expression for equivalent plastic strain increment. However, it should be thought of as a defined reference value without much physical interpretation.

5. Compute a number as given by

$$M = \frac{\Delta\bar{\varepsilon}}{\Delta\bar{\varepsilon}_{AL}} \tag{76}$$

where $\Delta\bar{\varepsilon}_{AL}$ is the allowable total strain increment and was selected as .0002 in/in in the present study. M is rounded off to the nearest integer value (greater than or equal to 1).

6. Divide total increments of strain by M.

7. For each subincrement of strain from 1 through M perform the following calculations:

   a. Compute $\bar{\sigma}$

$$\bar{\sigma} = [\sigma_s^2 + \sigma_\theta^2 - \sigma_s\,\sigma_\theta]^{1/2} \tag{77}$$

b. Compute $\{\frac{\partial \bar{\sigma}}{\partial \sigma}\}$

$$\frac{\partial \bar{\sigma}}{\partial \sigma_s} = \frac{1}{\bar{\sigma}} [\sigma_s - .5\, \sigma_\theta]$$
$$\frac{\partial \bar{\sigma}}{\partial \sigma_\theta} = \frac{1}{\bar{\sigma}} [\sigma_\theta - .5\, \sigma_s] \tag{78}$$

c. Based on the value of equivalent uniaxial strain, set H' in Eq. 61 equal to proper value.

d. Compute $d\bar{\varepsilon}^P$ by Eq. 60. If $d\bar{\varepsilon}^P < 0$, set $d\bar{\varepsilon}^P = 0$. Also, if $\bar{\sigma}$ is less than $\bar{\sigma}_{max}$ for all load increments, $d\bar{\varepsilon}^P = 0$.

e. Compute $d\varepsilon_s^P$, $d\varepsilon_\theta^P$ by Eq. 55.

f. Compute $d\sigma_s$, $d\sigma_\theta$ by Eq. 58.

g. Add increments of stress $d\sigma_s$, $d\sigma_\theta$, plastic strains $d\varepsilon_s^P$, $d\varepsilon_\theta^P$, $d\bar{\varepsilon}^P$, to previous values.

h. Return to step a for next subincrement of strain.

8. Compute partial derivatives of total strain with respect to each generalized coordinate for the element.

$$\frac{\partial \varepsilon_s}{\partial q_i} = \frac{\partial e_s}{\partial q_i} + e_{13} \frac{\partial e_{13}}{\partial q_i} + z \frac{\partial \chi_s}{\partial q_i}$$
$$\frac{\partial \varepsilon_\theta}{\partial q_i} = \frac{\partial e_\theta}{\partial q_i} + z \frac{\partial \chi_\theta}{\partial q_i} \tag{79}$$

It is noted that the partial derivatives computed in step A are used again.

9. If needed, compute contribution of this numerical integration station to pseudo plastic forces and add to previous value. From Eq. 48 the increment due to the station under consideration is

$$\Delta Q_i^P = \Delta z \; 2\pi L r \; W^z \; \lfloor \frac{\partial \varepsilon}{\partial q_i} \rfloor \; [D] \; \{\varepsilon^P\} \tag{80}$$

In this calculation the trapezoidal rule is used to integrate through the thickness. Thus $W^z = .5$ at $z = \pm \frac{T}{2}$ and 1.0 elsewhere.

10. If needed, compute contribution of station to plasticity stiffness matrix and add to previous contributions.
    a. Compute the matrix [C] as given by Eq. 62. If $d\bar{\varepsilon}^P$ for last subincrement in step 7 is zero, set [C] = [0]
    b. Compute contribution to plasticity matrix as given by Eq. 46 using numerical integration of the thickness.

$$\frac{\partial \Delta Q_i^P}{\partial q_j} = \Delta K_{ij}^P = -2\,\pi L r \;\Delta z \; W^z \;[(D_{11}\varepsilon_s^P + D_{12}\varepsilon_\theta^P)\,\frac{\partial e_{13}}{\partial q_i}\;\frac{\partial e_{13}}{\partial q_j}$$

$$- \lfloor \frac{\partial \varepsilon}{\partial q_i} \rfloor \; [D] \; [C] \; \{\frac{\partial \varepsilon}{\partial q_j}\}] \tag{81}$$

where the subscripts on D refer to elements of the [D] matrix.

11. Return to step 1 for next numerical integration station.

H. If needed, compute the pseudo forces due to geometric nonlinearities as given by Eq. 33.

$$Q_i^{NL} = -2\pi rLt \left[\frac{1}{2} e_{13}^2 \left(\frac{\partial e_s}{\partial q_i} D_{11} + \frac{\partial e_\theta}{\partial q_i} D_{21}\right) + e_{13} \frac{\partial e_{13}}{\partial q_i} \left(D_{11}\left(e_s + \frac{1}{2} e_{13}^2\right) + D_{12} e_\theta\right)\right] \qquad (82)$$

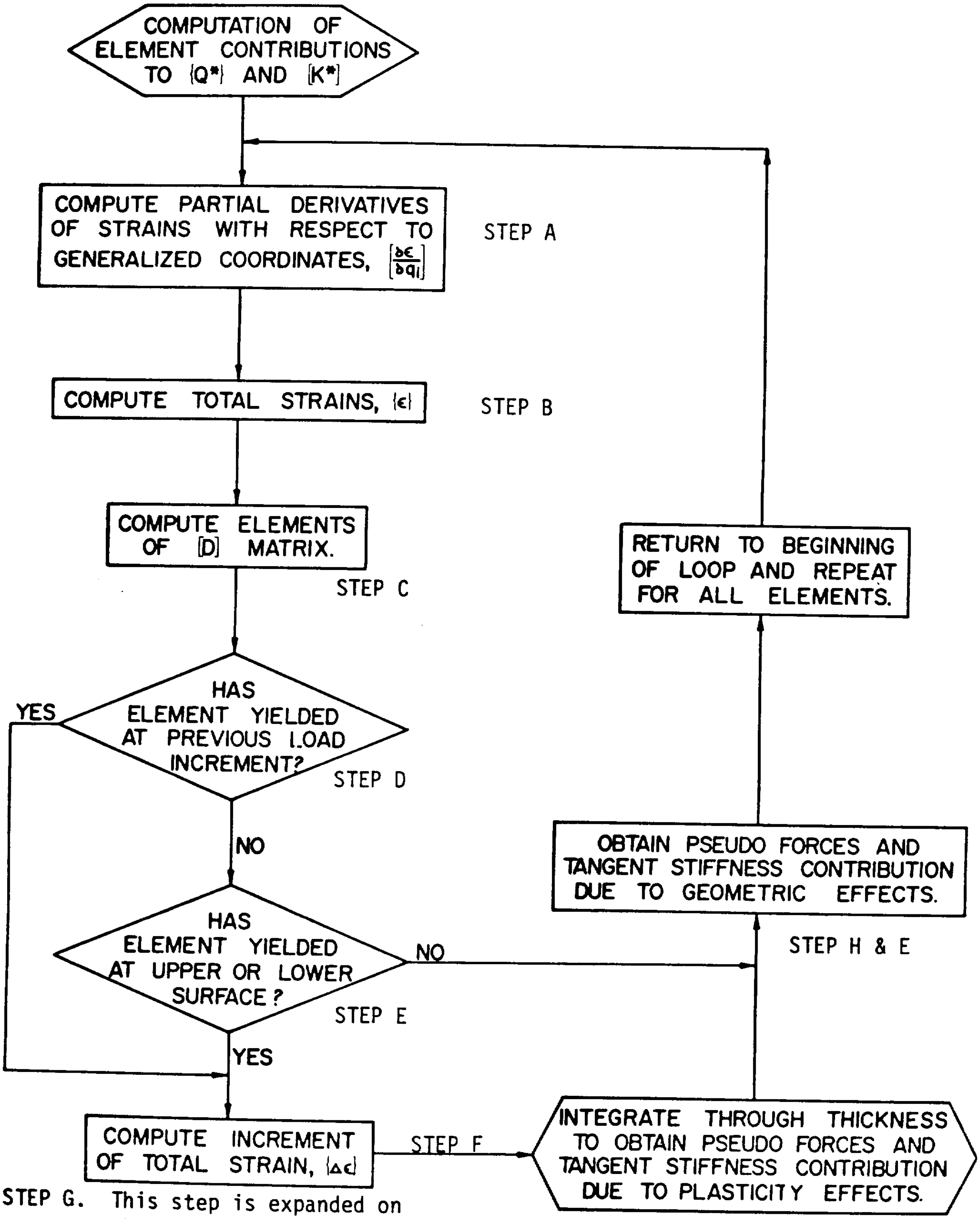
I. If needed, compute the contribution to the tangent stiffness matrix due to geometric nonlinearities as given by Eq. 47. The expansion for this term follows in the same manner as for Eq. 82 and is not given here.

The above computational algorithm is presented in flowchart form on the following pages.

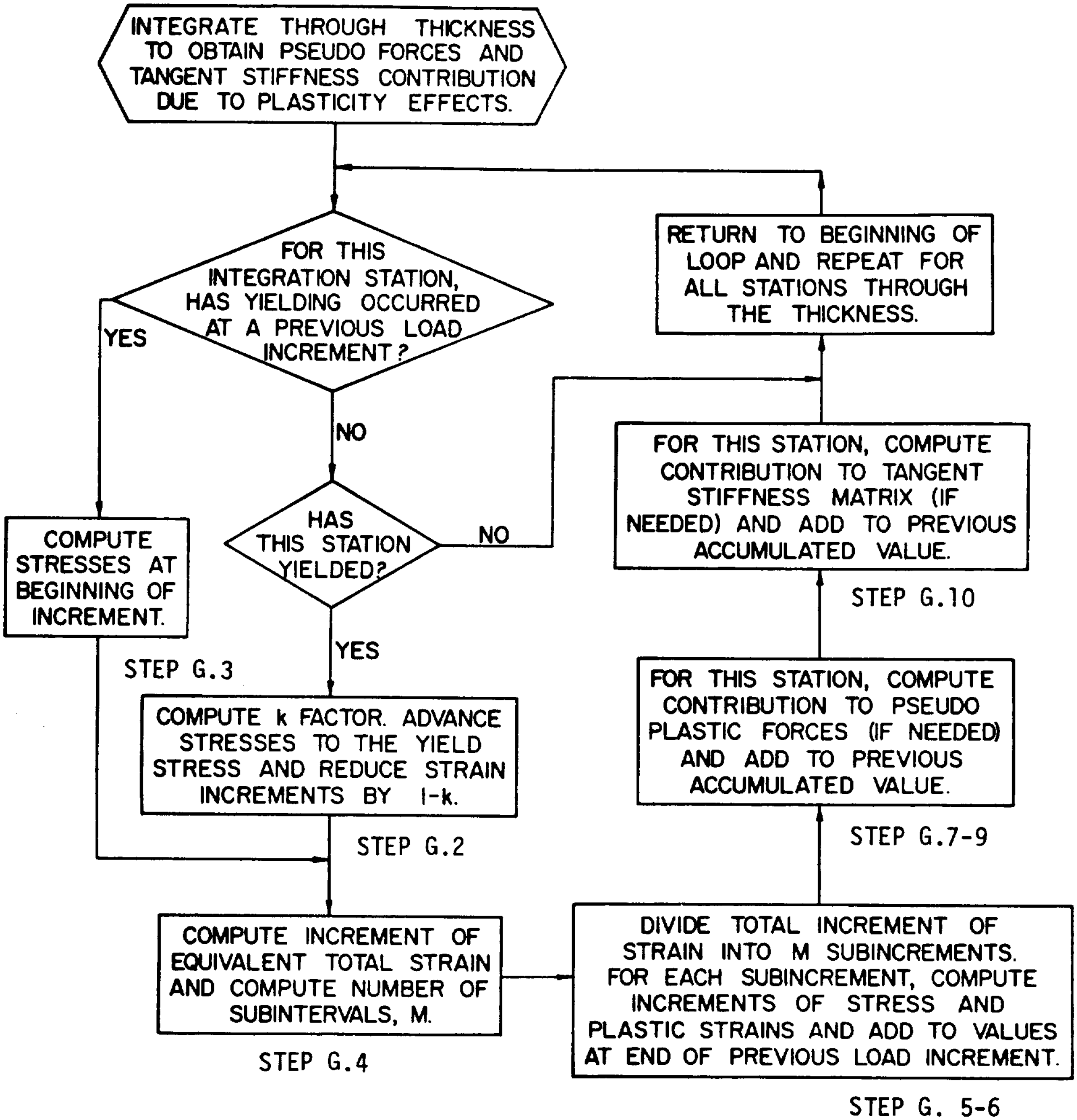
An attempt has been made in this section to present the details of the computational procedure. It should be noted that the partial derivatives computed in step A play a very important role in the calculations. Further if a higher order numerical integration formula were used along the length of the element, a procedure similar to the one just presented could be used with the partial derivative being evaluated at different stations along the length as well as the thickness. Furthermore, it should be observed that very little additional effort is needed to compute the plasticity matrix than is already required to compute the pseudo plastic forces.

The use of subincrements for the computation of stress and strain increments was first proposed by Huffington[44] to prevent obtaining a complex value for k. While this has not been a problem in the present research it was found that difficulties occur in staying on the assumed stress-strain curve unless the subincrements are used. It should be noted that a different number of subincrements are used for each element and each numerical integration station. The minimum number is of course one. Since the writer's original implementations of the above, it has been found that a very similar approach is proposed by Zienkiewicz and Nayak.[135]

COMPUTATIONAL FLOWCHART



STEP G. This step is expanded on the next page.


INTEGRATE THROUGH THICKNESS TO OBTAIN PSEUDO FORCES AND TANGENT STIFFNESS CONTRIBUTION DUE TO PLASTICITY EFFECTS.
FOR THIS INTEGRATION STATION, HAS YIELDING OCCURRED AT A PREVIOUS LOAD INCREMENT?
YES
NO
RETURN TO BEGINNING OF LOOP AND REPEAT FOR ALL STATIONS THROUGH THE THICKNESS.
FOR THIS STATION, COMPUTE CONTRIBUTION TO TANGENT STIFFNESS MATRIX (IF NEEDED) AND ADD TO PREVIOUS ACCUMULATED VALUE.
STEP G.10
HAS THIS STATION YIELDED?
NO
YES
COMPUTE STRESSES AT BEGINNING OF INCREMENT.
STEP G.3
COMPUTE k FACTOR. ADVANCE STRESSES TO THE YIELD STRESS AND REDUCE STRAIN INCREMENTS BY 1-k.
STEP G.2
FOR THIS STATION, COMPUTE CONTRIBUTION TO PSEUDO PLASTIC FORCES (IF NEEDED) AND ADD TO PREVIOUS ACCUMULATED VALUE.
STEP G.7-9
COMPUTE INCREMENT OF EQUIVALENT TOTAL STRAIN AND COMPUTE NUMBER OF SUBINTERVALS, M.
STEP G.4
DIVIDE TOTAL INCREMENT OF STRAIN INTO M SUBINCREMENTS. FOR EACH SUBINCREMENT, COMPUTE INCREMENTS OF STRESS AND PLASTIC STRAINS AND ADD TO VALUES AT END OF PREVIOUS LOAD INCREMENT.
STEP G. 5-6

CHECKOUT PROBLEMS

As is usual in the development of a computer program it is necessary to check the entire development by solving some typical problems and comparing the results with known solutions, solutions by other researchers, or experimental results. Two problems were chosen for this purpose.

The first is the large elastic-plastic deflection of a torispherical shell shown in Fig. 4 which has been analyzed previously by Yaghmai.[129] The geometrical dimensions of the shell are:

diameter of head skirt	=	100"
radius of sphere	=	100"
meridional radius of torus	=	20"
thickness	=	.8"

The material of the shell is elastic-perfectly plastic with a yield stress σ_y = 30,000 psi, Young's modulus E = 30 x 10^6 psi, and Poisson's ratio ν = .3.

Figure 4 presents the results from Ref. 129 for two different load increments using the tangent stiffness solution procedure (no correction term) and three sets of present results obtained by the Newton-Raphson, tangent stiffness, and first order self-correcting procedure. This latter procedure solves the equations of the form

$$[K + K^{NL} + K^{P}] \{\Delta q\} = \{\Delta P'\} + 1.2 \{f\} \tag{83}$$

The results from the Newton-Raphson procedure must be considered the correct solution for a load increment of 60 psi. However, it should be emphasized that the "converged" solution depends on the increment of load being used. From Fig. 4 it is rather obvious that the first order self-correcting procedure gives considerably better results that the purely incremental stiffness approach. Further the results are in reasonable agreement with the results reported by Yaghmai.[129]

The computer time used per load increment for each element was .09 sec (IBM 360/65) for the computational procedure presented herein as compared with .765 sec. (CDC 6400) reported by Yaghamai. The present results are for eleven stations through the thickness as compared with seventeen stations used in Ref. 129. However, the number of stations through the thickness has little effect on the computer time required in either procedure.

The second problem chosen to check the computer code is a circular, mild steel plate tested by Onat and Haythornthwaite.[91] The plate has a thickness of .19", a diameter of 6.375", and was loaded by a .5" diameter circular punch at the center of the plate. Young's modulus and Poisson's ratio were taken as 29×10^6 psi and .32 respectively. A uniaxial yield stress of 36,000 psi was used with perfectly plastic behavior up to an equivalent uniaxial plastic strain of .011 in/in. Next, a secondary modulus of 700,000 psi was used to an equivalent plastic strain of .05 in/in. For plastic strains larger than .05 in/in the material was assumed to be perfectly plastic. This stress-plastic strain behavior is shown in Fig. 5 which was plotted from the actual behavior of a numerical integration station near the applied load. It should be noted that Ref. 91 does not give the stress strain curve and thus the assumed behavior may be somewhat in error.

In this analysis the material underneath the .5" diameter punch was assumed to be rigid. Analyses were conducted using 25 and 11 elements and by several solution procedures.

The theoretical and experimental results for the load-deflection behavior are shown in Fig. 6. From this figure it can be seen that the theoretical results are above the experimental results. Undoubtedly part of this can be attributed to the assumption of rigid material underneath the punch. Considering this factor and the approximations in the stress strain curve, the correlation between theory and experiment is considered quite satisfactory. It is especially gratifying to observe that accurate results are obtained using only eleven elements.

Theoretical curves for the equivalent uniaxial plastic strain, meridional stress, and circumferential stress are shown in Figs. 7, 8, and 9, respectively. Excellent agreement is observed between the 25 and 11 element idealizations.

During the course of the study of the flat plate problem several interesting observations were made. First, it was found that 5 numerical integration stations were insufficient and in fact yielded meaningless results. This occured after all numerical integration stations had yielded on both sides of a node near the applied load. After considerable study it has been concluded that this observed behavior can be explained by Eq. 51 for the tangent stiffness matrix. When all elements have yielded the plastic strain increment for perfectly plastic material is almost equal to the total strain increment. Thus, the matrix [C] in Eq. 51 is almost an identity matrix. For this case, a singularity exists due to the process of numerical integration. The first term in Eq. 51 is integrated very accurately as previously described. However a five point integration through the thickness is rather crude and is believed to be the source of inaccuracy. The difficulty is removed by using 11 integration stations through the thickness. Eleven trapezoidal integration stations were used for all results reported herein. A comparison between trapezoidal and Simpson integration is given in a later section.

The second interesting observation in the influence of using subincrements to evaluate the increment of stresses and plastic strains. For the 300 pound increment Newton-Raphson solution it was found that as many as 85 subincrements were used for a station adjacent to the applied load. By forcing the procedure to use only a single subincrement the same Newton-Raphson procedure failed to converge in 10 iterations at 1200 lbs. The computer code automatically cut the increment by a factor of 4 (75 lbs) and continued until the specified value of 3000 lbs was reached. Thus, the use of subincrements influences the rate of convergence of the Newton-Raphson procedure. The deflection at the center for 3000 lbs. load was not appreciably different for the two cases but the one subincrement solution showed an equivalent uniaxial stress of over 100,000 psi which is quite disturbing.

SOLUTION PROCEDURES AND THEIR EVALUATION

Anyone who expects the authors to specify one particular procedure as being superior to all others will be disappointed. The objective of this section is to simply point out the advantages and shortcomings of each solution procedure. By performing this task in a systematic manner it is hoped that the reader may gain insight into what procedure is best suited for his formulation. The present section presents the solution procedures according to class with cross reference of the more efficient procedures being presented in the next section.

Test Problem

As the objective of this research is to test many different solution procedures, it was necessary to select a test problem which does not require excessive computer time and at the same time demonstrates the desired behavior. For these reasons the eleven element circular plate was chosen as the test problem. The eleven elements starting at the edge of the load have equal lengths of .267". Eleven numerical integration stations were used through the thickness. The yield stress of the first two elements adjacent to the load were raised to 100,000 psi and 50,000, respectively. The shape of the stress-plastic strain curve is that given in Fig. 5 with a vertical translation of the curve for the first two elements. The total load was selected as 6000 lbs. but run for larger values in some cases.

The load deflection curve for this test case is shown in Fig. 10 as obtained by the Newton-Raphson procedure and by the first order self-correcting procedure (Eq. 83). The two numbers within the parentheses indicate the number of iterations required and the number of stations unloading, respectively. The converged solution for a load increment of 300 lbs. is of course the Newton-Raphson solution, but it will be observed later that the self-correcting incremental solution is closer to the true solution obtained with load refinement. The Newton-Raphson procedure was used with a load increment of 150 lbs. and found to be closer to the incremental solution. By coincident the error in the self-correcting incremental procedure for a 300 lb. load increment is compensated for by the error in computing the plastic strains.

It is noted that a small degree of unloading (7 stations) occurs in this problem. It was hoped that a more pronounced unloading would occur to serve as a critical test for the solution procedures.

Figures 11 and 12 present the meridional and circumferential stress respectively in the upper surface vs the distance from the center of the plate. It is observed that the values of the stresses are in better agreement than the displacements. While this is not the usual case in finite element work it is not unrealistic for almost perfectly plastic materials as used herein.

The movement of the stresses around the yield surface for two numerical integration stations which unload is shown in Figs. 13 and 14. It is noted that the stress behavior cannot be approximated to any satisfactory degree by proportional loading. Thus, deformation theories of plasticity would be completely inappropriate for the present problem. The x's in Figs. 13 and 14 go with the circles adjacent to them. The Newton-Raphson solutions are not presented above 5400 lbs. due to convergence problems and will be discussed in the next section. The points outside the original yield surface are correct and indicate appreciable strain hardening.

Based on the results presented in this section for the values of stresses and deflections it was decided to use the accuracy of the load-deflection curve as a means of evaluating the different solution procedures. The curve referred to as the converged solution is the solution which converged with load increment refinement and was obtained by two different solution procedures.

Exact Solution, $f = 0$

The first class of solution procedures is that in which the equations of equilibrium are satisfied exactly. Here the function f is defined as:

$$\{f\} = -[K]\{q\} + P\{\bar{P}\} + \{Q^{NL}\} + \{Q^{P}\} \tag{84}$$

where $\{Q^{NL}\}$ = pseudo forces due to geometric nonlinearities

$\{Q^{P}\}$ = pseudo force due to material nonlinearities.

The first procedure tested was the Newton-Raphson procedure. For this procedure the load is increased in increments and a value for the first guess of the displacements is obtained by some extrapolation procedure. For these values of the displacements the function $\{f(q_0)\}$ is computed. After this a value of $\{\Delta q\}$ is sought such that $\{f(q_0 + \Delta q)\} = \{0\}$. Expanding this latter expression into a first order Taylor's series yields the expression for $\{\Delta q\}$.

$$([K] + [K^{NL}] + [K^{P}])\{\Delta q\} = \{f(q_0)\} \tag{85}$$

where

$$[K^{NL}] = -[\frac{\partial Q_i^{NL}}{\partial q_j}] \tag{86}$$

and

$$[K^P] = -[\frac{\partial Q_i^P}{\partial q_j}] \tag{87}$$

$\{\Delta q\}$ is determined by solving Eq. 85 and the next guess for $\{q\}$ is $\{q_0 + \Delta q\}$ The force unbalance $\{f\}$ and matrices $[K^{NL}]$ and $[K^P]$ are updated and a new value of $\{\Delta q\}$ is determined by solving Eq. 85 again. The solution process is continued until hopefully the process converges.

The results obtained for the test problem by the Newton-Raphson procedure are presented in Fig. 10. However, it is noted that the maximum load obtained is 5400 lbs. At this load the procedure failed to converge in 10 iterations. The load increment was automatically reduced twice by a factor of 4 and each time failed to converge in ten additional iterations.

The Newton-Raphson procedure was the first procedure tested and consequently a considerable effort was expended to overcome the convergence difficulties. The following were tried:

1. Under-relaxation. An under-relaxation factor was applied to the displacement increments at each cycle of iteration. The relaxation factor was reduced by .2 (from a value of 1.0) every three cycles with the minimum value being restricted to 0.5. This process yielded one additional increment of load after the load increment had been reduced by a factor of 4. Thus for practical purposes the process did not yield any significant improvement in convergence characteristics.

2. Second order extrapolation - For most cases the first guess was

obtained using a linear extrapolation of the displacement. A quadratic extrapolation procedure was used and it was found that the process converged in fewer iterations. However, it did not converge for a load higher than 5400 lbs.

3. Reduction of load increment - The load increment was reduced to 150 lbs. and a second order extrapolation procedure was used for the first trial values. This yielded converged results for a load of 5,750 lbs. However, the procedure failed to converge for higher loads.

4. Modified Newton Raphson - The procedure was modified so that the nonlinear stiffness matrix and plasticity matrix were updated at the beginning of the increment and only every 4 cycles thereafter. This process shortened the computer time required but again the process failed to converge for a load above 5400 lbs.

Close examination of the iterational process reveals that the trial values oscillate between values which cause loading and unloading respectively. The process does not diverge but simply oscillates.

Referring back to the section on computational procedure it is observed that the matrix $[K^P]$ can have discrete discontinuities when unloading occurs. Since all the convergence proofs for the Newton Raphson procedure assume a continuous first derivative, it is not possible to state specifically that the procedure will converge when there is a possibility of loading with elastic unloading. Furthermore under such conditions there is no unique solution for the deflections. All these considerations lead to the conclusion that additional refinements are needed in the Newton Raphson solution procedure to make it applicable to problems involving complex combined loading-unloading behavior.

The very simplest form of the modified Newton-Raphson procedure is obtained when the matrices $[K^{NL}]$ and $[K^{P}]$ are set equal to zero. For this case the process reduces to the method of successive approximations as given by Eq. 3. This method was tried on the test problem to only a limited degree as it is a special form of the Newton-Raphson procedure. Results by successive approximations indicated that the convergence rate is very slow for large plastic strains and that convergence does not occur for elastic unloading. The convergence rate was so very slow in some cases that the difference between successive values of the deflections was so small as to indicate false convergence. The slow convergence rate has previously been pointed out in Ref. 128.

Incremental Stiffness, $\dot{f} = 0$

As will be shown in this section a special case of $\{\dot{f}\} = \{0\}$ is the incremental stiffness method which has been so popular in nonlinear analyses. However, most researchers in nonlinear analyses have already realized that considerably better results are obtained when the first order self-correcting form is used. Consequently, only a short discussion will be presented for $\{\dot{f}\} = \{0\}$.

Taking the derivative of Eq. 84 with respect to the scaler load parameter P yields

$$[K]\,\{\dot{q}\} = \{\bar{P}\} + \{\dot{Q}^{NL}\} + \{\dot{Q}^{P}\} \qquad (88)$$

where the dot indicates differentiation with respect to P.

The simplest solution procedure for Eq. 88 is an Euler forward difference for $\{\dot{q}\}$ and an Euler backwards difference for $\{\dot{Q}^{NL}\}$ and $\{\dot{Q}^{P}\}$. This form may be used for moderately nonlinear problems but a rather exhaustive study by Haisler[34] has shown that such a representation of $\{\dot{Q}^{NL}\}$ is very unstable. For this reason no studies were conducted for $\{\dot{f}\} = \{0\}$ with all pseudo forces on the right hand side.

Applying chain rule differentiation to $\{\dot{Q}^{NL}\}$ and to $\{\dot{Q}^{P}\}$ two other forms of Eq. 88 are obtained.

$$([K] + [K^{NL}] + [K^{P}])\ \{\dot{q}\} = \{\dot{\bar{P}}\} \tag{89}$$

$$([K] + [K^{NL}])\ \{\dot{q}\} = \{\dot{\bar{P}}\} + \{\dot{Q}^{P}\} \tag{90}$$

Equation 89 is a first order nonlinear differential equation which may be solved by any one of a large number of solution procedures such as predictor-corrector and Runge-Kutta methods. The only method considered herein is the simple Euler forward difference which reduces to the incremental stiffness approach. Results for this method applied to the test problem with a load increment of 300 lbs. are shown in Fig. 15. It is noted that the theoretical solution tends to "drift" appreciably from the converged solution.

Also shown in Fig. 15 are the results for the solution of Eq. 90 using an Euler forward difference for $\{\dot{q}\}$ and a backwards difference for $\{\dot{Q}^{P}\}$. This solution procedure is quite inaccurate for a load increment of 300 lbs.

The remaining result in Fig. 15 is for the solution of Eq. 90 with $\{\dot{Q}^{P}\}$ being determined by a linear extrapolation procedure. This procedure improves the accuracy of the load deflection curve; but close examination of the data reveals that an unrealistic elastic unloading is obtained. The same type of extrapolation procedure was tried with the first order self-correcting form but is not presented due to erroneous results for unloading.

First Order Self-Correcting, $\dot{f} + Zf = 0$

This particular solution procedure is referred to as a first order self-correcting solution procedure. Applying this formula to the definition of {f} as given by Eq. 84 and using chain rule differentiation of both $\{\dot{Q}^{NL}\}$ and $\{\dot{Q}^{P}\}$ yields

$$([K] + [K^{NL}] + [K^{P}])\ \{\dot{q}\} = \{\dot{\bar{P}}\} + Z\{f\} \tag{91}$$

Equation 91 is a first order nonlinear differential equation which may be solved by any one of a large number of solution procedures. The only procedure considered here is the Euler forward difference expression for $\{\dot{q}\}$. For this procedure Eq. 91 reduces to:

$$([K] + [K^{NL}] + [K^{P}])\ \{\Delta q\} = \{\Delta P'\} + Z\Delta P\{f\} \tag{92}$$

Comparing Eq. 92 with Eq. 85 for the Newton-Raphson procedure reveals a great degree of similarity. In particular for $Z\Delta P = 1.0$ the right hand side represents the unbalance in force due to $\{\Delta P'\}$ and the total unbalance at the beginning of the load increment. This has led to the naming of the procedure as an incremental procedure with a one step Newton-Raphson correction.[119] Experiences for $Z\Delta P = 1$ have shown a considerable improvement over the purely incremental approach. However for large load increments the procedure still tends to "drift".

The above explanation may be used to develop an intuitive feel for the proper selection of $Z\Delta P$. In the first place since $\{f\}$ is the value of the force unbalance at the beginning of the increment, then $\{f\}$ at the end of the increment will be slightly larger and thus a factor of $Z\Delta P > 1$ is indicated. However if the Newton-Raphson approach is converging by oscillating about the true solution it is clear that a large value of $Z\Delta P$ would cause the solution procedure to diverge. Based on this argument and experience a value of $Z\Delta P$ of 1.2 to 1.3 is believed to be conservative from a stability point of view and to increase the accuracy somewhat. This logic was discussed at a recent conference with Levine[61] who decided to conduct a numerical evaluation of using $Z\Delta P > 1$. He reports no appreciable difference between results for $Z\Delta P = 1$ and $Z\Delta P = 1.3$. Similar recent studies by the present authors seems to confirm the conclusions reached by Levine. These results are presented in a later section of this report. Thus, in summary there are some doubts about the advantages using $Z\Delta P > 1$ but little doubt about the increased accuracy obtained in going from $Z\Delta P = 0.0$ (incremental stiffness) to $Z\Delta P = 1.0$.

Results for the load-deflection curve of the test problem with $Z\Delta P = 1.2$ are presented in Fig. 16. The results show that good accuracy is obtained using only ten load increments and for all practical purposes the results have converged for 20 load increments.

As the formulation given by Eq. 92 is self-correcting the idea arises that it is not necessary to update $[K^{NL}]$ and $[K^{P}]$ every load increment. This was studied with the results being presented in Fig. 17. It is noted from Fig. 17 that 40 total increments with 13 updates in the coefficient matrix does not change the accuracy appreciably. However, comparing the results in Fig. 16 for 10 increments with the results in Fig. 17 with 13 updates reveals comparable degrees of accuracy. These results indicate that it does not make much difference in accuracy whether one uses a small number of increments and updates every time or uses a large number of increments and only updates so that the total number is about the same.

Another form of $\dot{f} + Zf = 0$ is obtained by using pseudo forces for $\{\dot{Q}^P\}$ and the nonlinear stiffness matrix approach for $\{\dot{Q}^{NL}\}$. Again the pseudo force approach was not attempted for $\{\dot{Q}^{NL}\}$ as previous studies have indicated numerical instability problems. Using a forward difference for $\{\dot{q}\}$ and a backwards difference for $\{\dot{Q}^P\}$ yields

$$([K] + [K^{NL}])\ \{\Delta q_i\} = \{\Delta P'\} + \{\Delta Q^P_{i-1}\} + Z\Delta P\{f\}_{i-1} \tag{93}$$

The test problem results using $Z\Delta P = 1.3$ are shown in Fig. 18 for three different load increments. The more refined results agree with those obtained through Eq. 92. Figure 18 reveals that reasonably accurate results are obtained when 60 increments of load are used. Figure 19 shows how the results are influenced by not updating $[K^{NL}]$ every load increment. It is noted that only the values at large values of the load are changed. This occurs simply because geometric nonlinearities are not significant for small deflections.

A comparison of the accuracy obtained through Eqs. 92 and 93 is shown in Fig. 20 for 20 load increments. It is observed that the results obtained through Eq. 93 are not nearly as good as the results obtained using Eq. 92. For this reason the solution for combined geometric and material nonlinearities would appear to best be achieved through Eq. 92.

This of course depends on the formulation as in some formulations it is very tedious to compute $[K^P]$. For nonlinearities due only to plastic deformations the solution through Eq. 93 appears to be very promising as it involves only a single inversion of the matrix [K].

Second Order Self-Correcting, $\ddot{f} + C\dot{f} + Zf = 0$

This particular form, which is called the second order self-correcting form, is a result of a long and tedious search for a method of solution for geometric nonlinearities which requires only a single "inversion" of the stiffness matrix and is very economical on computer storage requirements. It is a natural extension of the first order self-correcting form and is very appealing since the equation is the same as for damped harmonic motion.

Two different forms of this equation have been explored. In the first (Ref. 120) the expression for f (Eq. 84) is substituted into the equation $\ddot{f} + C\dot{f} + Zf = 0$ and certain terms collected to yield the equation for damped harmonic motion in terms of the displacement q. This form is then solved by an implicit four point backwards difference formula for the displacements as a function of the load P. The resulting solution oscillates about the true solution for geometric nonlinearities but these oscillations are undesirable for material nonlinearities.

The second form[76] of the equation is obtained as follows and was studied herein. The particular form is for C = 0. First Eq. 84 is rearranged in the form

$$[K]\,\{q_i\} = P_i\{\bar{P}\} + \{Q^P_{i-1}\} + \{Q^{NL}_{i-1}\} - \{f_i\} \qquad (94)$$

It is noted that the pseudo force terms are used as their known values at i-1. The unbalance in force f_i is determined from the equation.

$$\{\ddot{f}\} + Z\{f\} = 0 \tag{95}$$

which is

$$\{f\} = \{A\} \cos\sqrt{Z}\, s + \{B\} \sin\sqrt{Z}\, s \tag{96}$$

where s has the range from 0.0 to ΔP. The matrices of coefficients $\{A\}$ and $\{B\}$ are determined from the known values of $\{f\}$ and $\{\dot{f}\}$ at the beginning of the increment.

$$\{f(s = 0)\} = \{A\} = - [K]\{q_{i-1}\} + P_{i-1}\{\bar{P}\} + \{Q^{P}_{i-1}\} + \{Q^{NL}_{i-1}\} \tag{97}$$

$$\{\dot{f}(s = 0)\} = \sqrt{Z}\{B\} = - [K]\{\dot{q}_{i-1}\} + \{\bar{P}\} \tag{98}$$

The derivatives of $\{Q^P\}$ and $\{Q^{NL}\}$ are omitted in Eq. 98 as they are held constant over the increment as seen in Eq. 94.

Substituting the values for $\{A\}$ and $\{B\}$ given by Eqs. 97 and 98 into Eq. 96, using a backwards difference for $\{\dot{q}\}$, and using the results in Eq. 94 yields the following recurrsion relation for $\{q_i\}$ in terms of known quantities.

$$\{q_i\} = A_1\{q_{i-2}\} + A_2 \{q_{i-1}\} + [K]^{-1} (A_3\{\bar{P}\} + A_4 \{Q^{NL}_{i-1}\} + A_5 \{Q^{P}_{i-1}\}) \tag{99}$$

where

$$A_1 = - \sin (\sqrt{Z}\, \Delta P)/(\sqrt{Z}\, \Delta P)$$

$$A_2 = \cos(\sqrt{Z}\, \Delta P) + \sin (\sqrt{Z}\, \Delta P)/(\sqrt{Z}\, \Delta P)$$

$$A_3 = P_i - \cos(\sqrt{Z}\,\Delta P)P_{i-1} - \sin(\sqrt{Z}\,\Delta P)/\sqrt{Z} \tag{100}$$

$$A_4 = +1. - \cos\sqrt{Z}\,\Delta P$$

$$A_5 = A_4$$

It should be noted that the application of the recurrence relation given by Eq. 99 requires only the symmetric stiffness matrix and 6 one-dimensional arrays.

The maximum value of P is chosen to be 100 and Z is varied in accordance to:

$$Z = \frac{2.0}{\Delta P\sqrt{P\Delta P}} \tag{101}$$

Results for the test problem using the second order self correcting form are shown in Fig. 21. It is noted that 200 load increments are needed to give a reasonably accurate solution. However, only a single "inversion" of the symmetric stiffness matrix is required.

It should be pointed out that the stability of this procedure is controlled appreciably by the parameters C and Z in addition to the step size ΔP. Massett and Stricklin[76] have indicated that a small positive value of C inhances the stability but causes the response to be somewhat "sluggish". The possibility exists of using a small negative value of C in conjunction with a small value of Z as given by Eq. 101. It would appear that the negative C would increase the response yet not be unstable. Hopefully the small value of Z would retain the stability. This approach is possible since the load increment ΔP traverses only a small portion of the complete response history. This is, of course, one of several possibilities that should be investigated.

DISCUSSION OF SOLUTION PROCEDURES

In this section the more promising solution procedures for geometric and/or material nonlinearities are summarized. However, before summarizing these procedures based on the studies presented herein, it should be emphasized that there is ample room for further studies. In particular the forms for both the first and second derivative formulations should be studied. Furthermore, the solution procedures presented herein should be subjected to a more severe test, e.g. when a large percentage of the structure transcends from a state of loading to elastic unloading.

Geometric Nonlinearities

For problems involving geometric nonlinearities there are three procedures worth discussing. The first is the modified Newton-Raphson procedure[35] which is the only method capable of yielding the exact solution. This procedure requires some updating of $[K^{NL}]$ and a very large number of iterative cycles where $\{Q^{NL}\}$ is evaluated each time.

The second method is the second order self-correcting form as given in Refs. 76 and 120. This procedure is very efficient on storage requirements and involves only a single "inversion" of the stiffness matrix. However, a large number of load increments must be used for accurate results which implies many calculations of the pseudo force terms $\{Q^{NL}\}$.

The third method is the first order self correcting form.[35] This procedure has previously been evaluated for $Z\Delta P = 1.0$ and was found to require many increments of load to achieve a converged solution. As each increment requires the forming of $[K^{NL}]$ and the solution of a system of equations the method required considerable computer time. To show the effects of using $Z\Delta P > 1.0$ a shallow cap under a point load at the apex was analyzed in the present study. The results are presented in Fig. 22. The shallow cap was analyzed using a $Z\Delta P$ of 1.0, 1.1, and 1.3. The results for $Z\Delta P$ of 1.0 and 1.1 remained stable while the value of 1.3 began to oscillate beyond a load of 50 pounds.

This particular problem illustrates that the value of $Z\Delta P$ which may be used depends on the shape of the load deflection curve. For stability problems where the load deflection curve has a horizonal tangent or for the test problem a value of 1.3 may be used with a reasonable amount of safety. But, for a load deflection curve which shows a softening followed by a rapid change to stiffening behavior, the value of $Z\Delta P > 1.0$ can give rise to numerical instabilities.

<u>Material Nonlinearities</u>

For material nonlinearities only, the consensus of opinion of the writers is that the first order self-correcting procedure is quite outstanding. Both forms as given by Eqs. 92 and 93 yield good results. Equation 93 is particulararily appealing as a simple recursion relation may be developed. This recursion relation is given by:

$$\{q_{i+1}\} = B_1\ \{q_i\} + [K]^{-1}\ (B_2\ \{\bar{P}\} + B_3\ \{Q_i^P\} + B_4\ \{Q_{i-1}^P\}) \qquad (102)$$

where $B_1 = 1 - Z\Delta P$

$$B_2 = \Delta P(1 + ZP_i) \qquad (103)$$

$$B_3 = 1 + Z\Delta P$$

$$B_4 = -1$$

The recursion relation given by Eq. 102 conserves computer storage space, requires one "inversion" of the matrix [K], and yields good results for a relatively small number of load increments as can be seen in Fig. 18. The solution procedure as given by Eq. 92 requires fewer load increments but requires the computation of $[K^P]$ and the solution of a set of algebraic equations for each load increment.

Combined Geometric and Material Nonlinearities

Two solution procedures are suited for the analysis of combined geometric and material nonlinearities. The first is the first order self-correcting procedure in the form given by Eq. 92. The other procedure is the second order self-correcting procedure using the recurrence relation given by Eq. 99.

Comparing the results in Figs. 16 and 21 reveals that comparable accuracy is obtained using 10 increments in the first order self-correcting or 200 increments in the second order self-correcting procedures. The required number of increments for comparable accuracy would be shifted considerably for problems involving predominatedly geometric nonlinearities. The shift would be to the benefit of the second order self-correcting procedure.

Simpson vs Trapeziodal Integration

For all the results presented in this report trapeziodal integration was used through the thickness with eleven integration stations being used. This was thought at first to be the best choice since the derivative of the plastic strain may be discontinuous through the thickness. However, in an attempt to improve the computation efficiency the relative merits of Simpson's vs. trapeziodal integration was studied with the results being presented in Figs. 23 and 24. It is clear from an examination of these figures that Simpson's integration is clearly superior to the trapeziodal integration rule. A simular conclusion has previously been reached by Krieg[53], who also tested higher order integration formulas. The plate considered in the example was the original test plate of Onat and Haythornthwaite.[91]

CONCLUSIONS

The primary conclusion to be reached from this study is that due to the efforts of a large number of people it is now possible to include nonlinear effects in stress analysis without the expenditure of excessive amounts of computer time. With further developments in solution procedures and formulation it should be possible to reduce the required computer time further. The efficiency of these computational procedures is evidenced by the relatively short run times - on the order of one minute on the IBM 360/65 for the problems studied herein. The program is written entirely in FORTRAN IV and readily adaptable to most computers.

In the past, most nonlinear solutions have been restricted to specific problems (plate with circular hole, etc.) and very few attempts have been made to solve large-scale problems involving both geometric and material nonlinearities. For the few existing computer codes which attempt to solve large-scale problems, the computer run times are excessive with run times of four hours being quoted. It is the writer's opinions that the time has come to utilize recent advances as presented herein and elsewhere for the development of efficient computer programs. The research group at Texas A&M University has already moved in this direction by developing codes for the static and dynamic analysis of shells of revolution under asymmetrical mechanical and thermal loads including both geometric and material nonlinearities.

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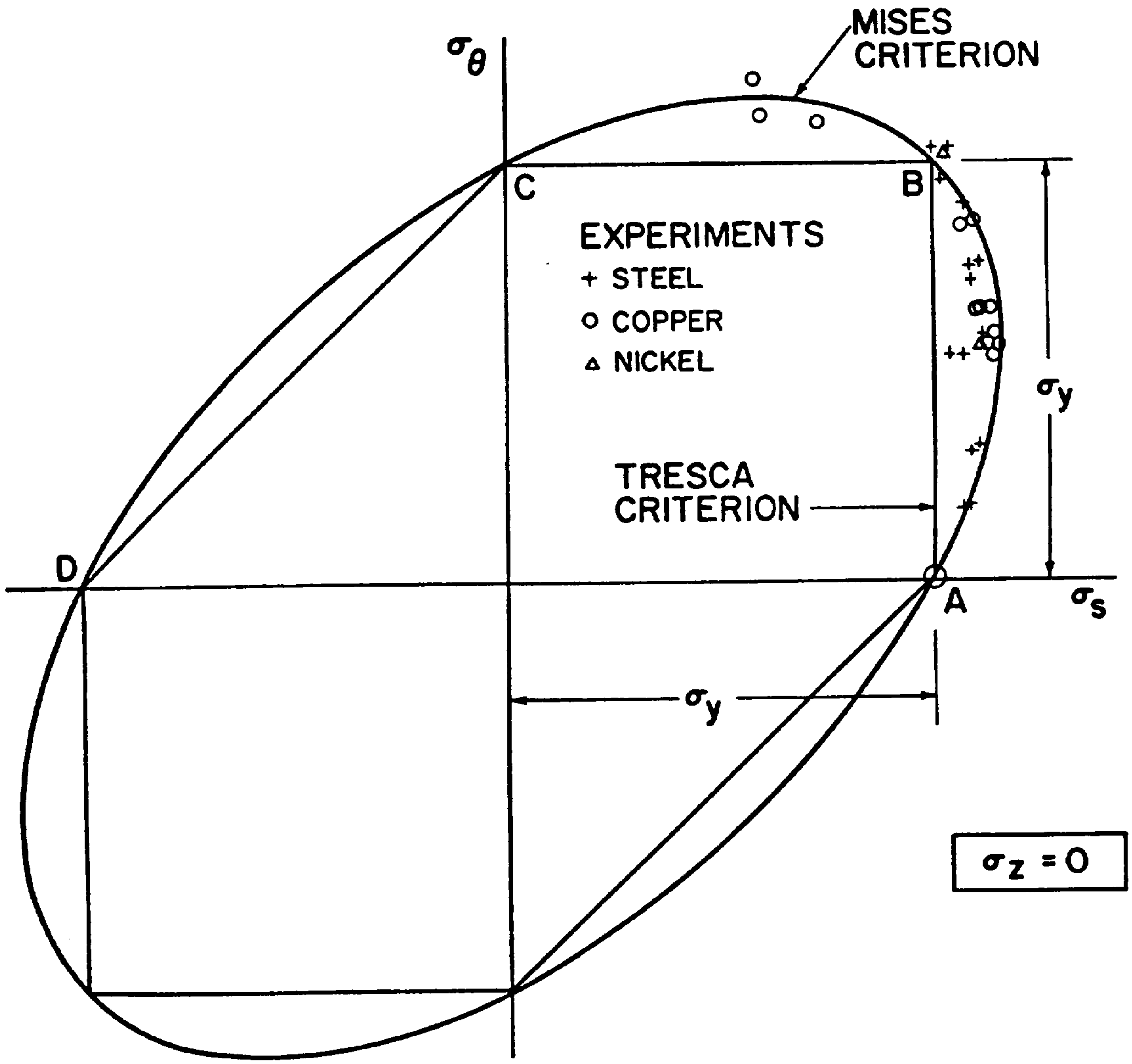

FIG. 1 YIELD CRITERION

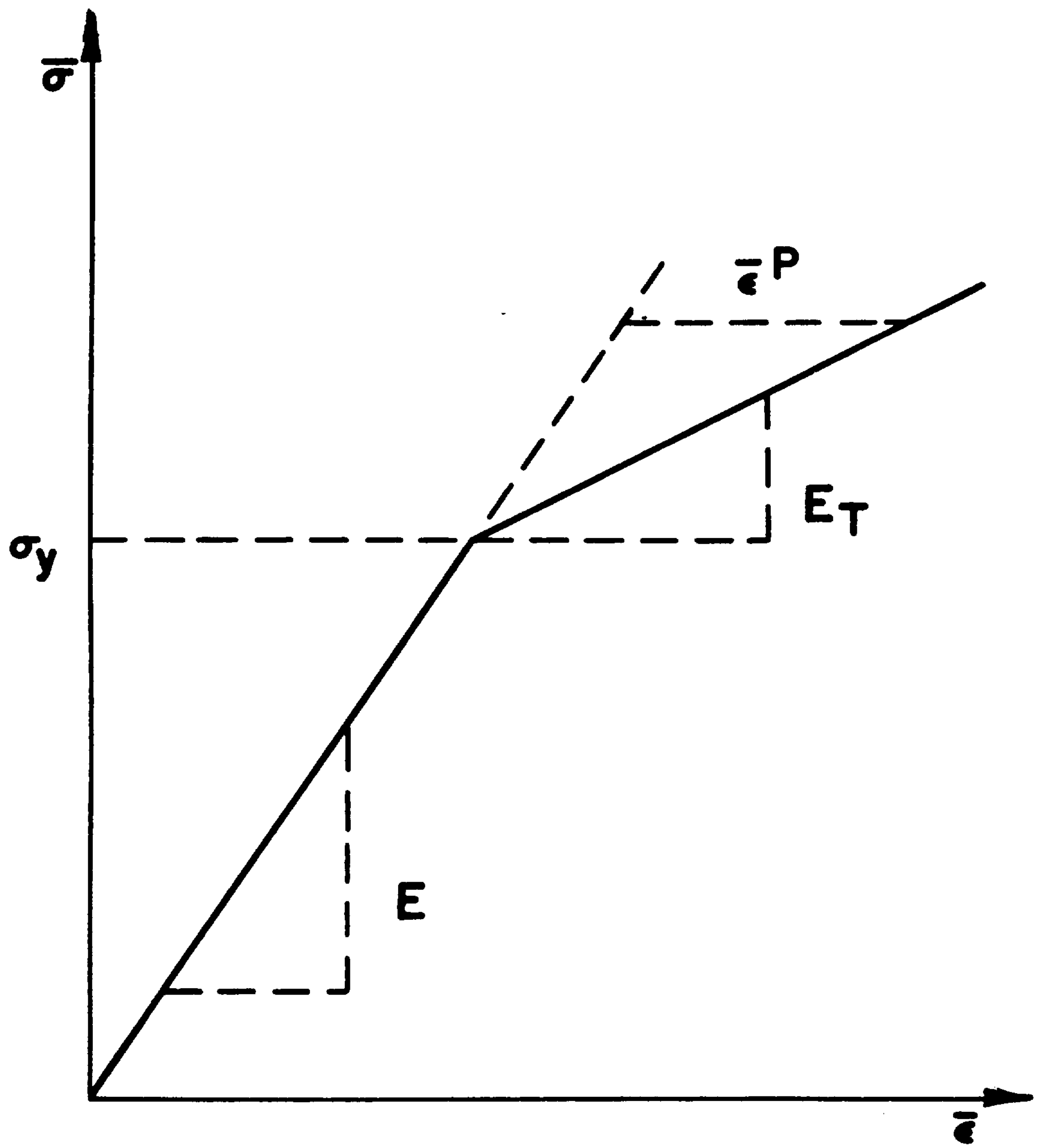

FIG. 2 PIECEWISE LINEAR UNIAXIAL STRESS-STRAIN CURVE

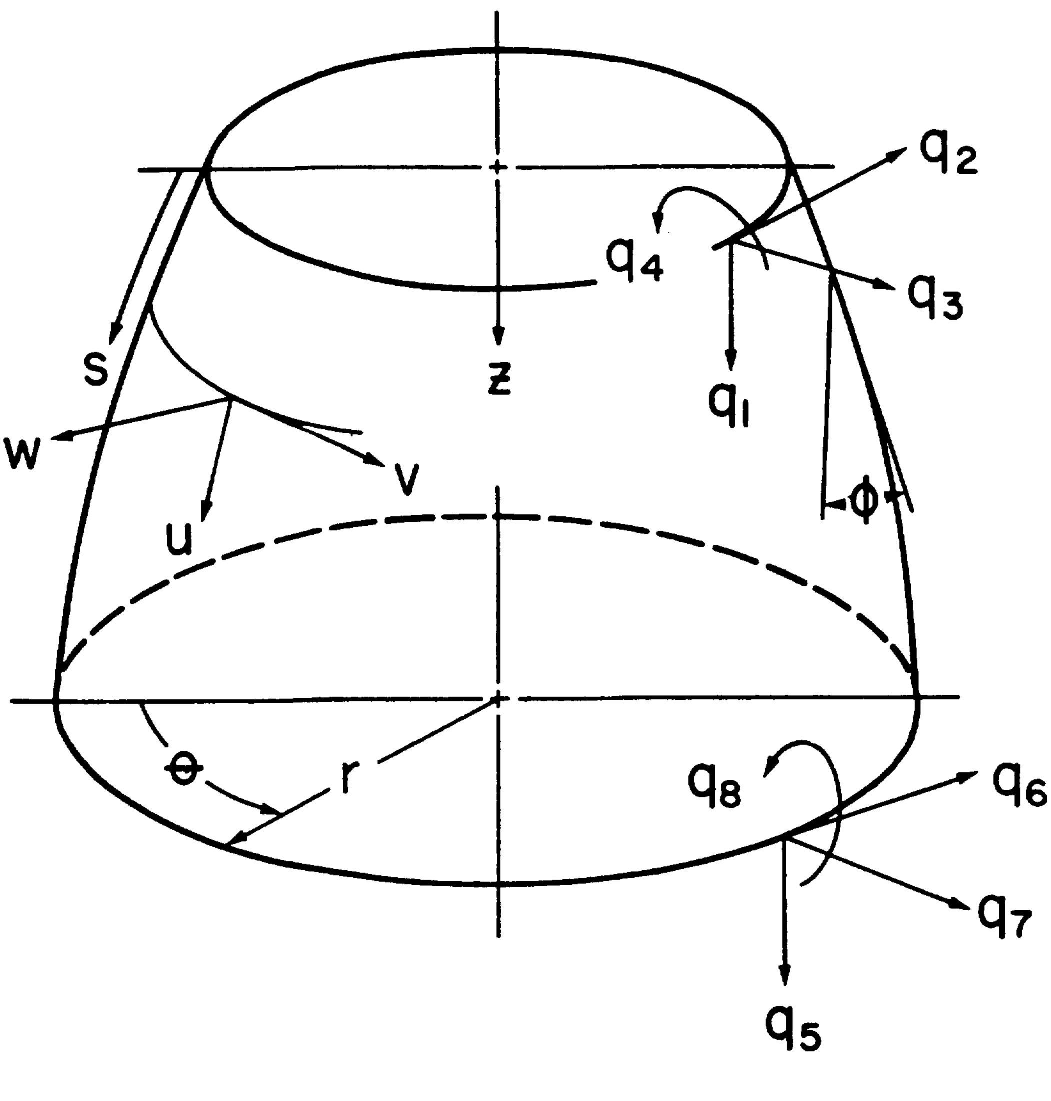

FIG. 3 GENERALIZED COORDINATES OF SHELL ELEMENT

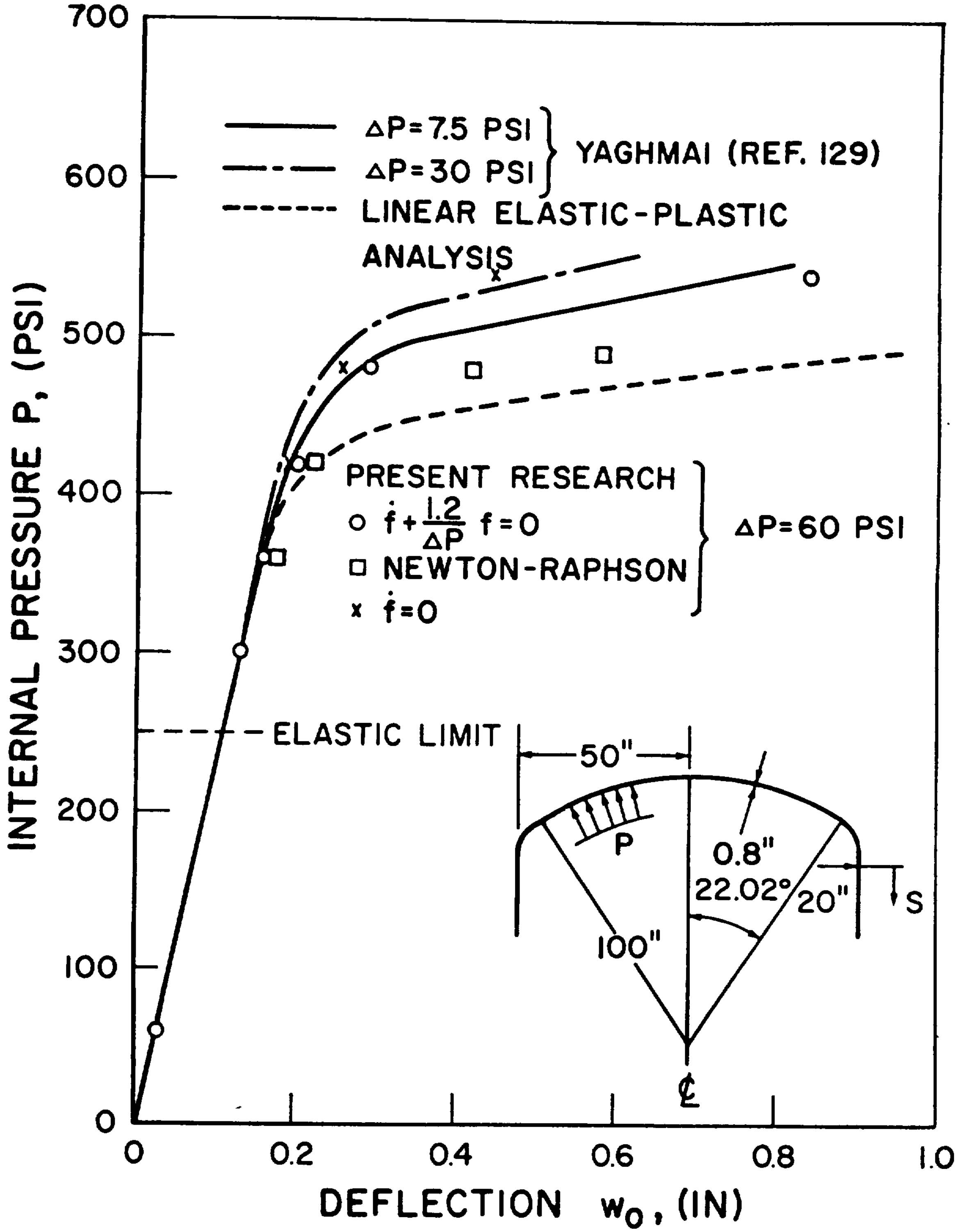

FIG. 4 THE NORMAL DEFLECTION AT THE APEX, w_0

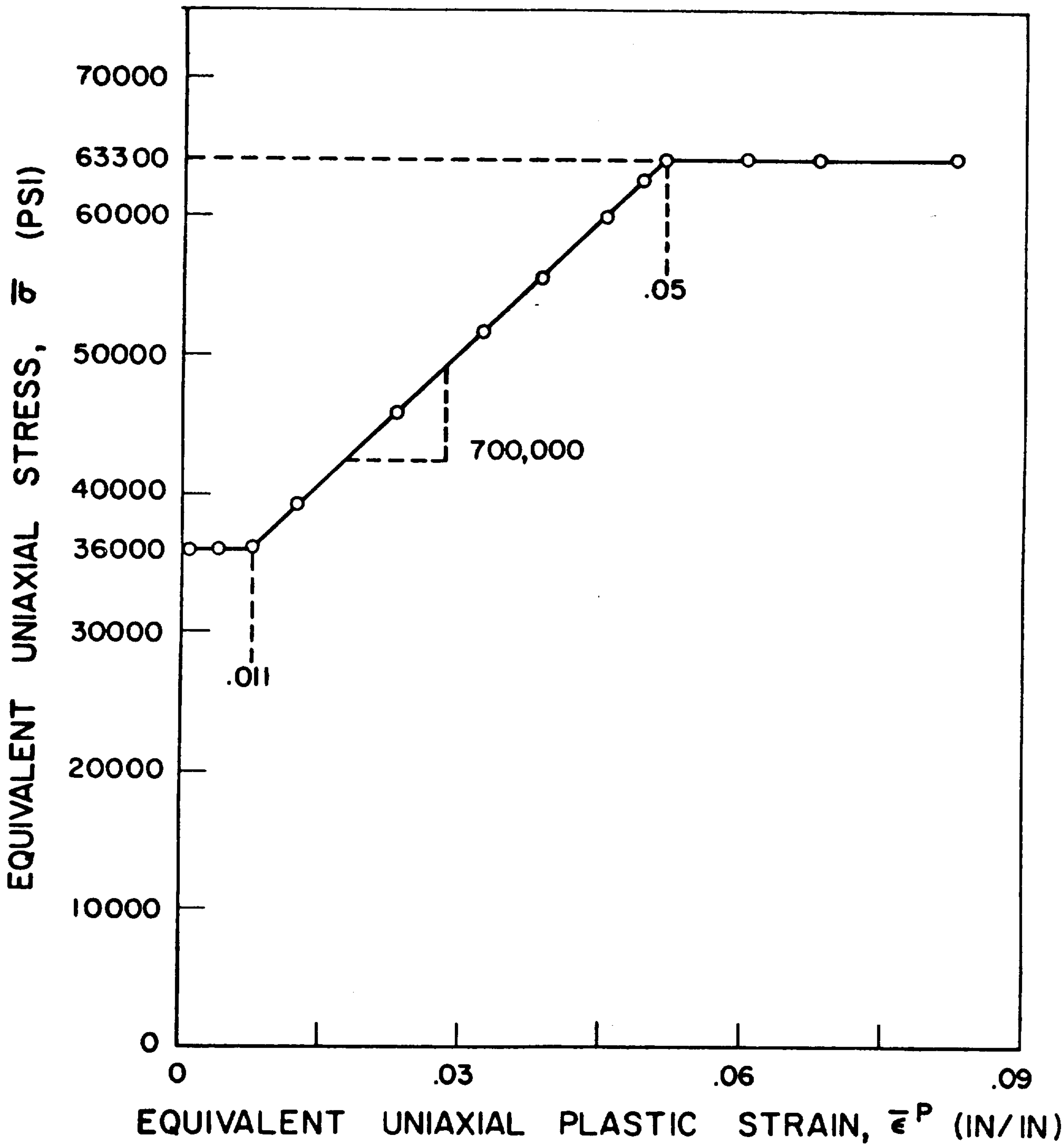

FIG. 5 EQUIVALENT STRESS–STRAIN CURVE AFTER YIELDING

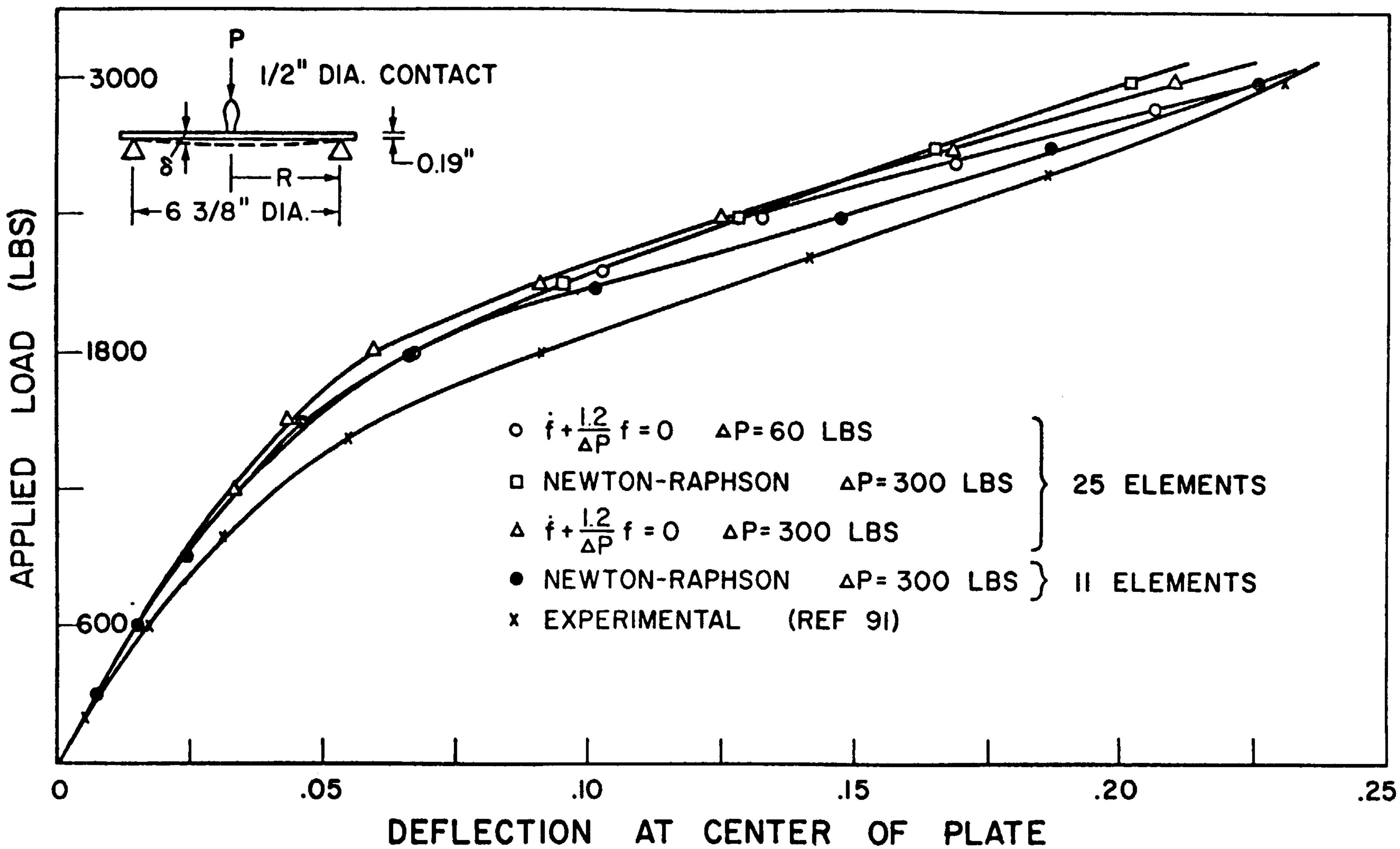

FIG. 6 EXPERIMENTAL AND THEORETICAL LOAD-DEFLECTION CURVE FOR CIRCULAR PLATE

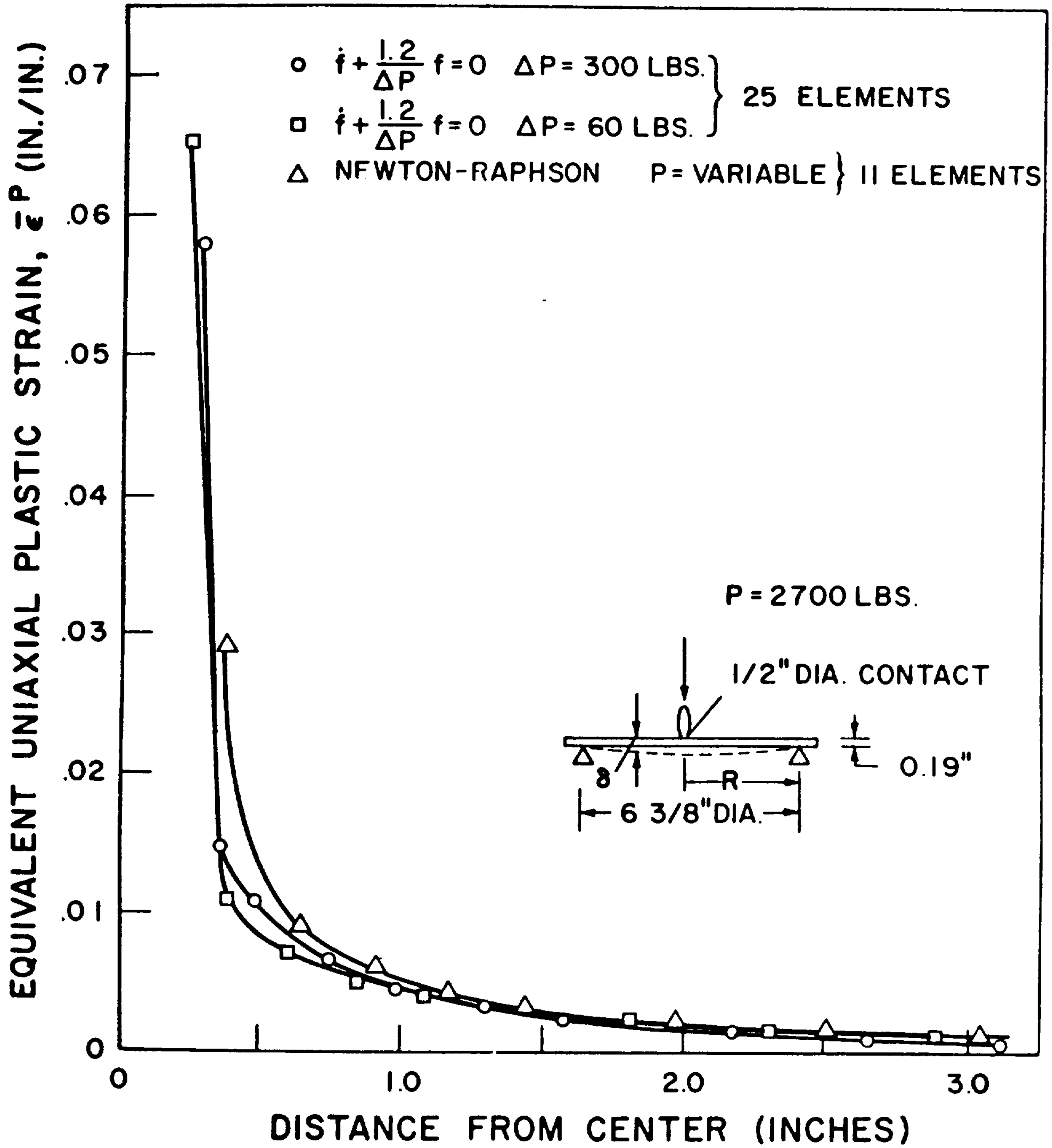

FIG. 7 EQUIVALENT PLASTIC STRAIN VS. DISTANCE FROM CENTER FOR FLAT PLATE FOR P=2700 LBS.

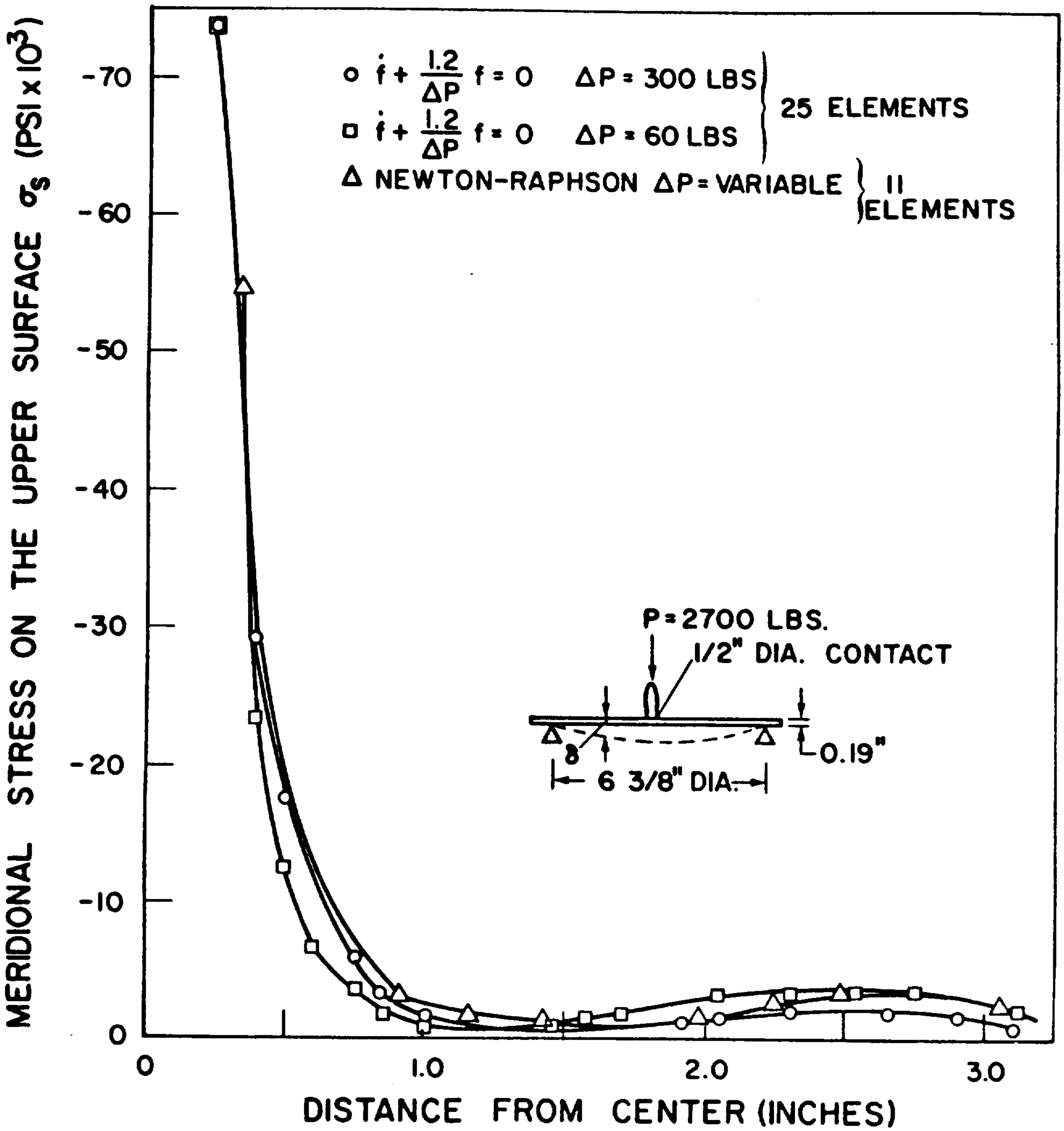

FIG.8 MERIDIONAL STRESS AT UPPER SURFACE vs. DISTANCE FROM CENTER (P= 2700 LBS)

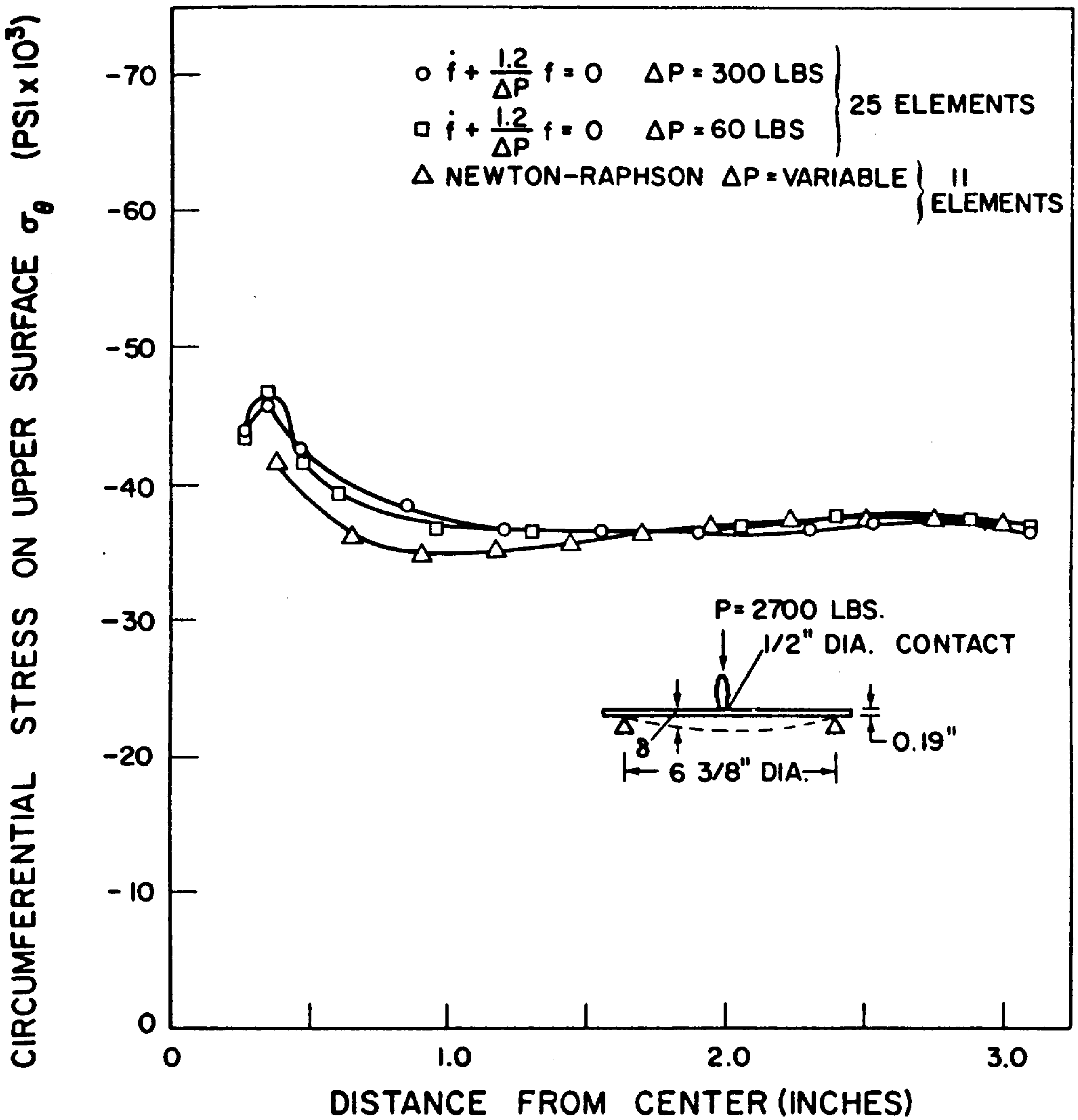

FIG.9 CIRCUMFERENTIAL STRESS AT UPPER SURFACE vs. DISTANCE FROM CENTER (P=2700 LBS)

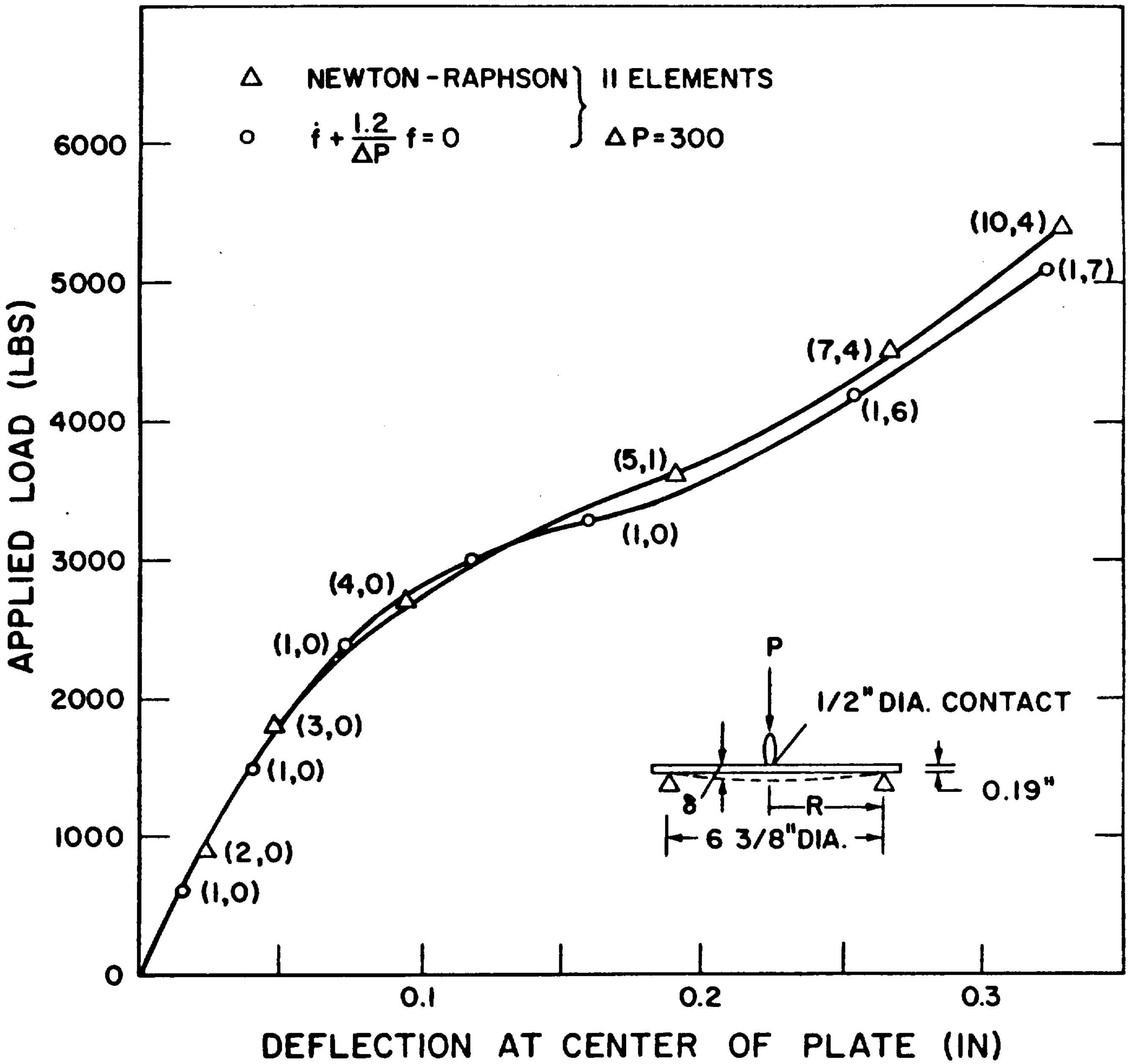

FIG. 10 LOAD DEFLECTION CURVE FOR FLAT PLATE

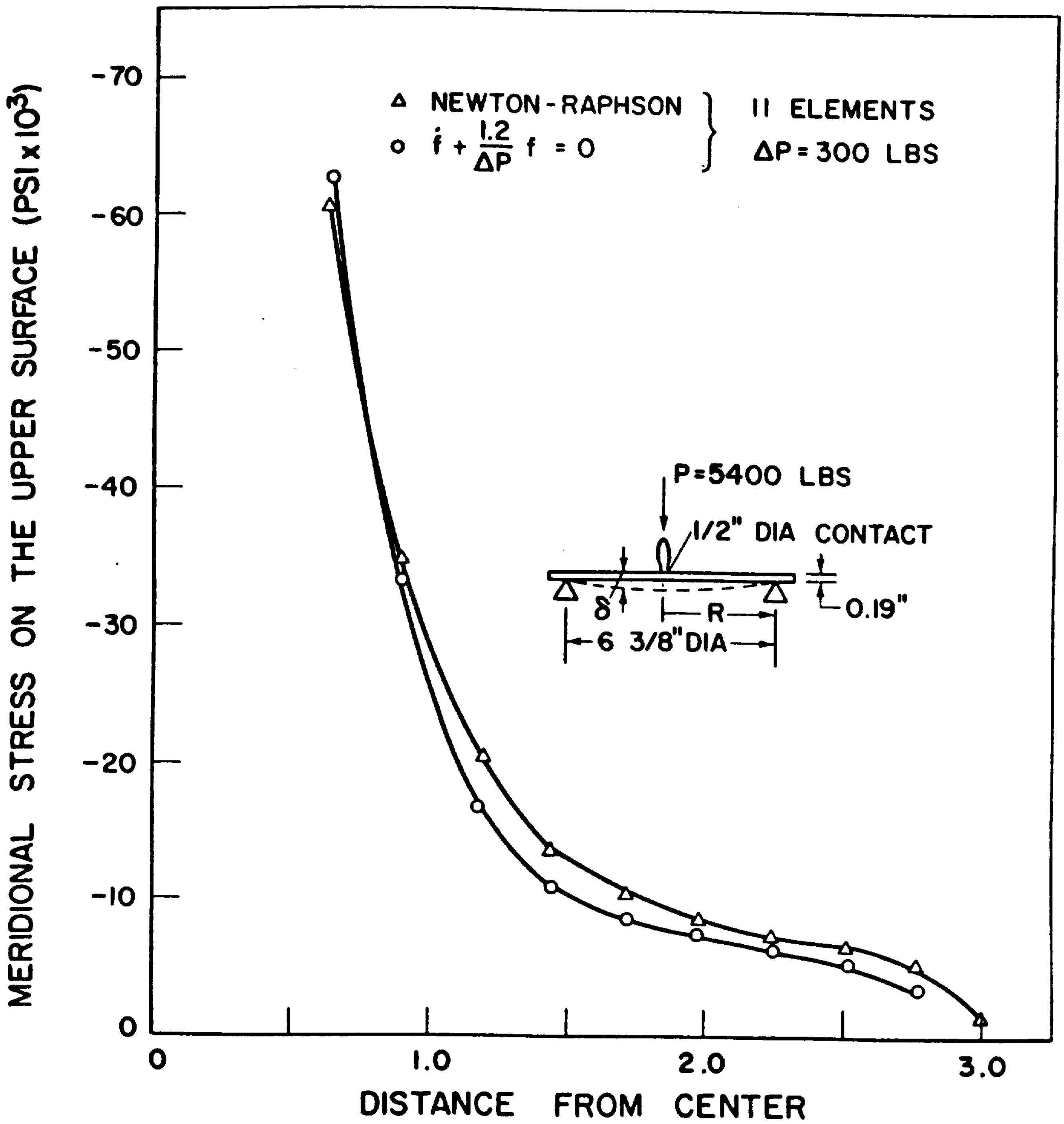

FIG. 11 MERIDIONAL STRESS AT UPPER SURFACE vs. DISTANCE FROM CENTER OF PLATE FOR 5400 LBS

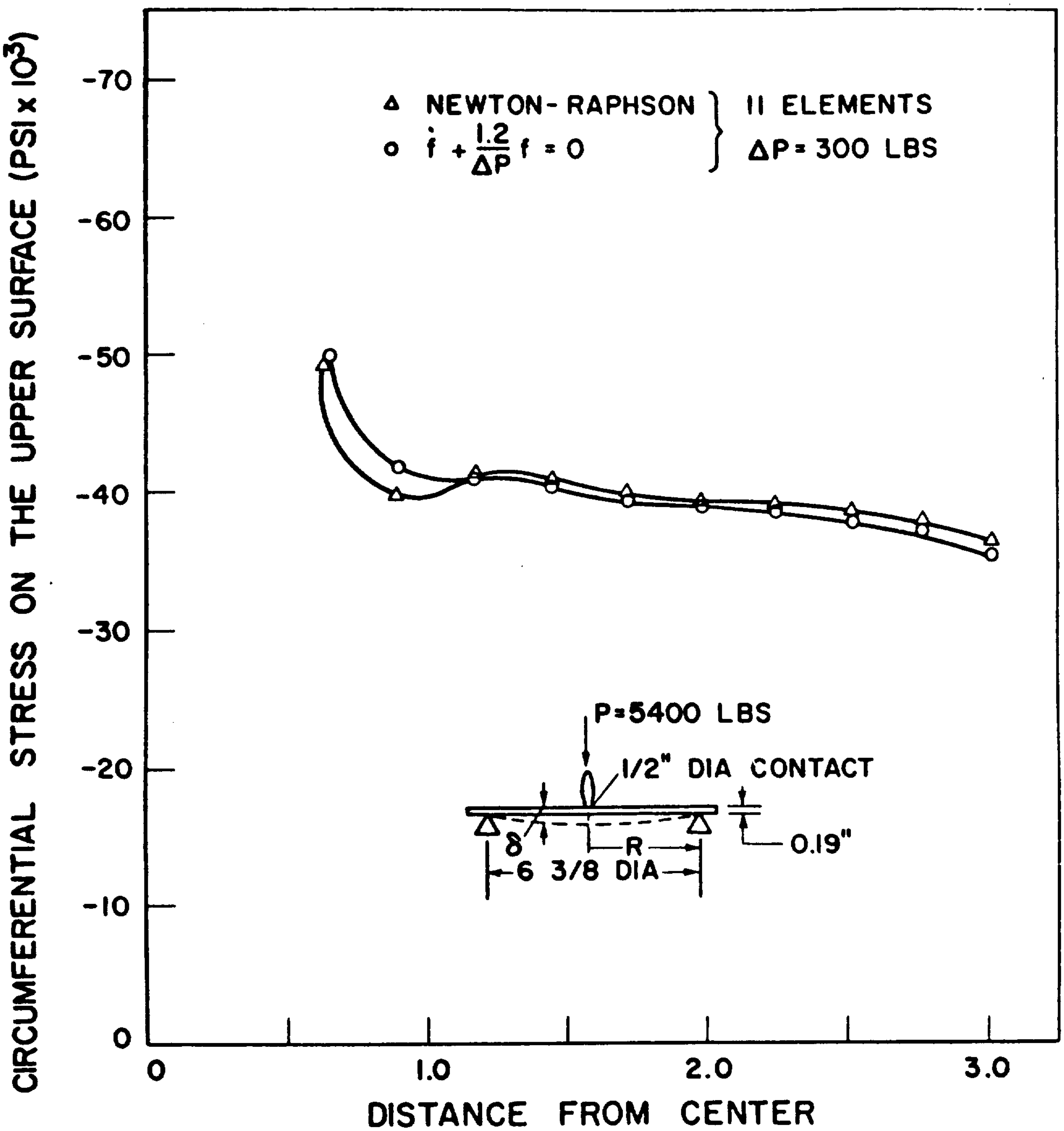

FIG. 12 CIRCUMFERENTIAL STRESS AT UPPER SURFACE vs. DISTANCE FROM CENTER OF PLATE FOR 5400 LBS

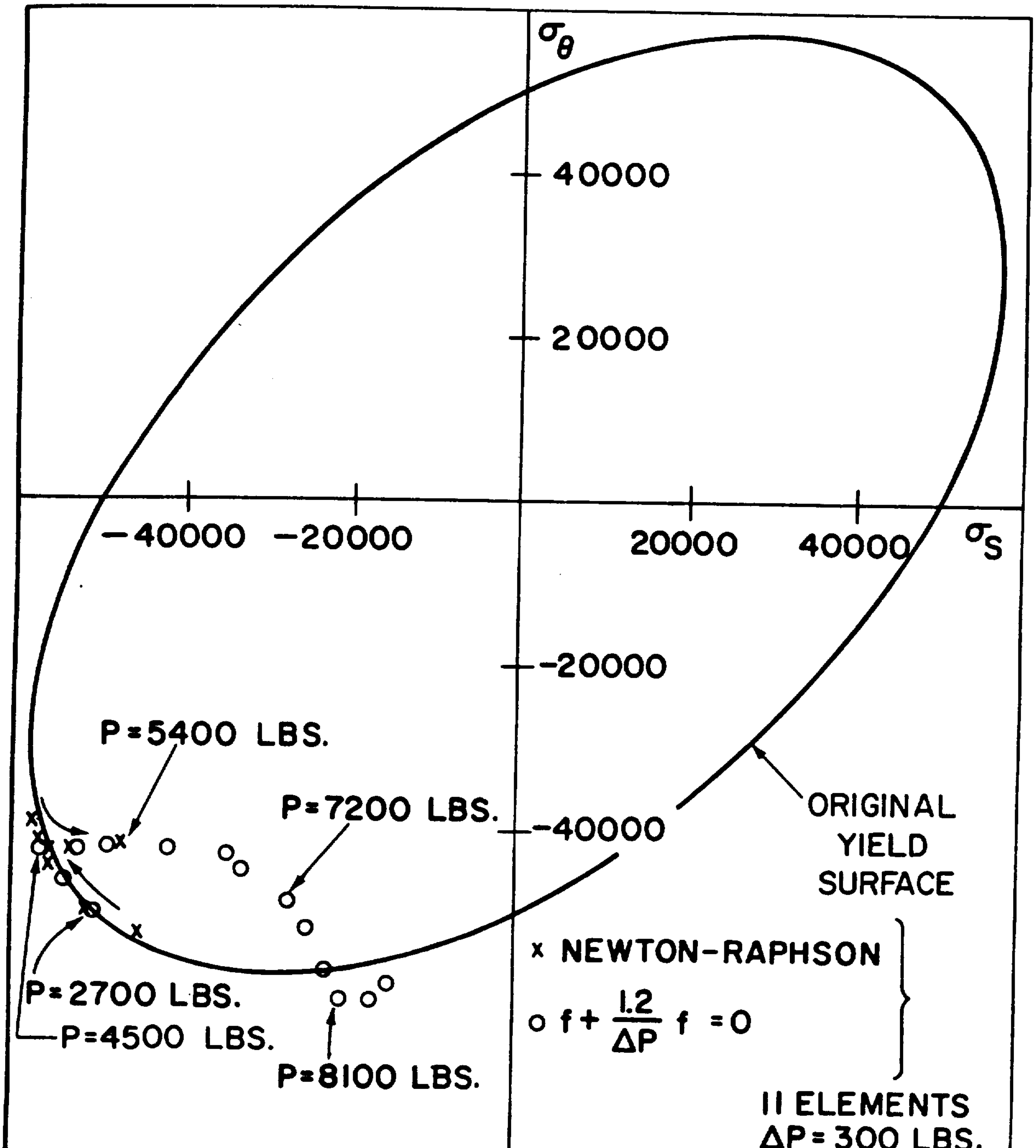

FIG. 13 YIELD SURFACE BEHAVIOR FOR Z=.057" AT .65" FROM CENTER

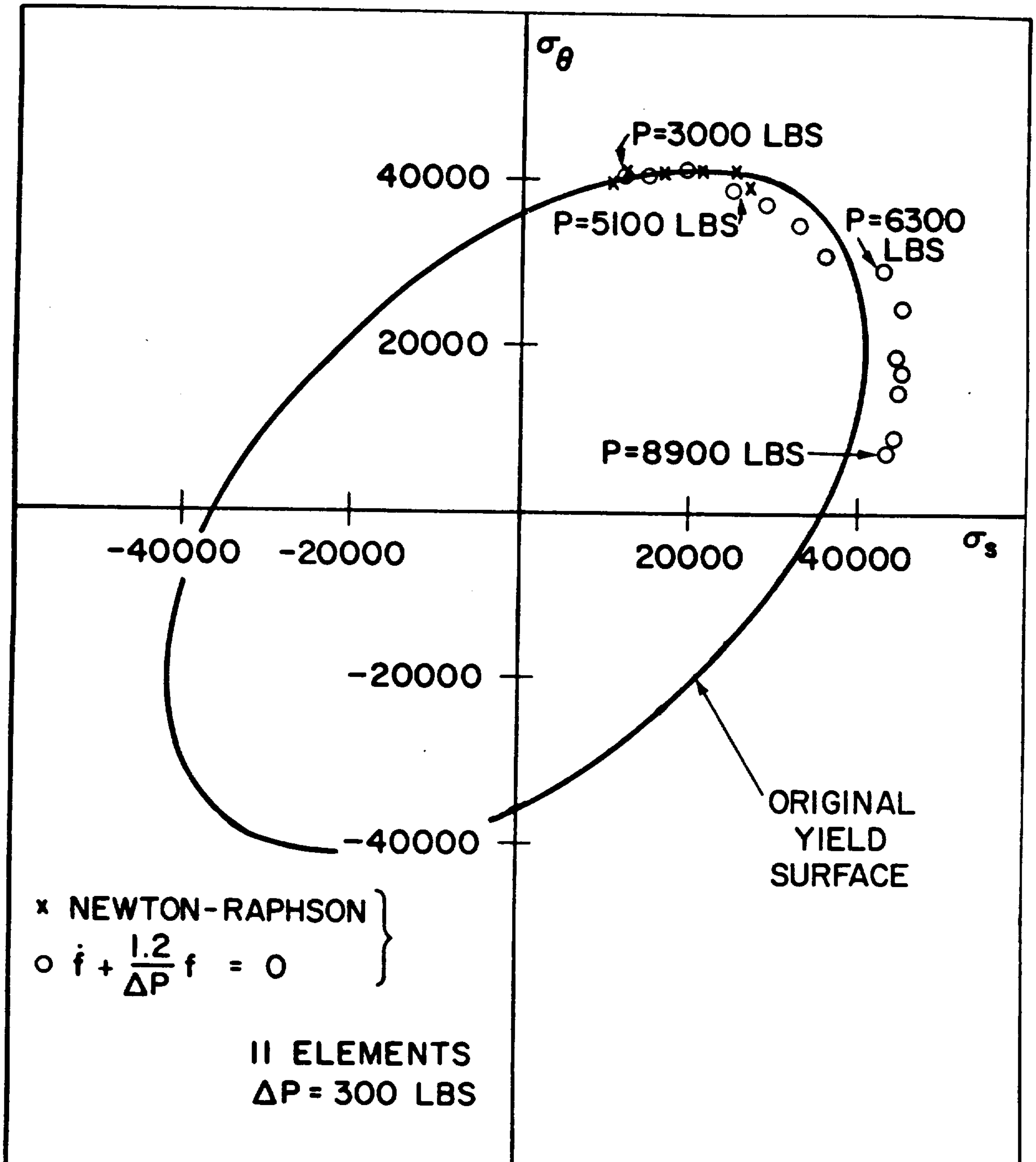

FIG. 14 YIELD SURFACE BEHAVIOR FOR LOWER SURFACE AT 2.52" FROM CENTER

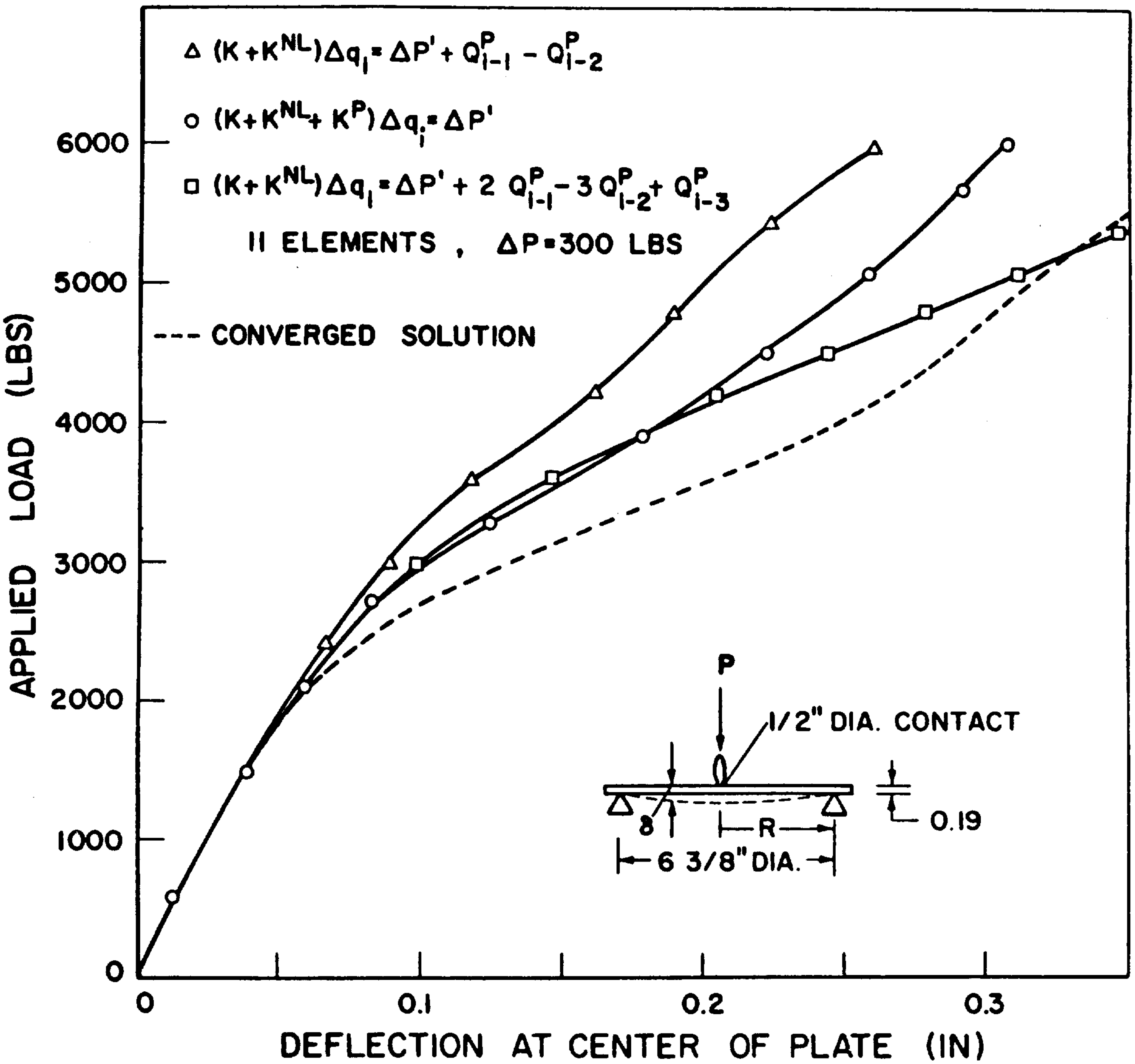

FIG. 15 LOAD-DEFLECTION CURVES FOR $\dot{f}=0$

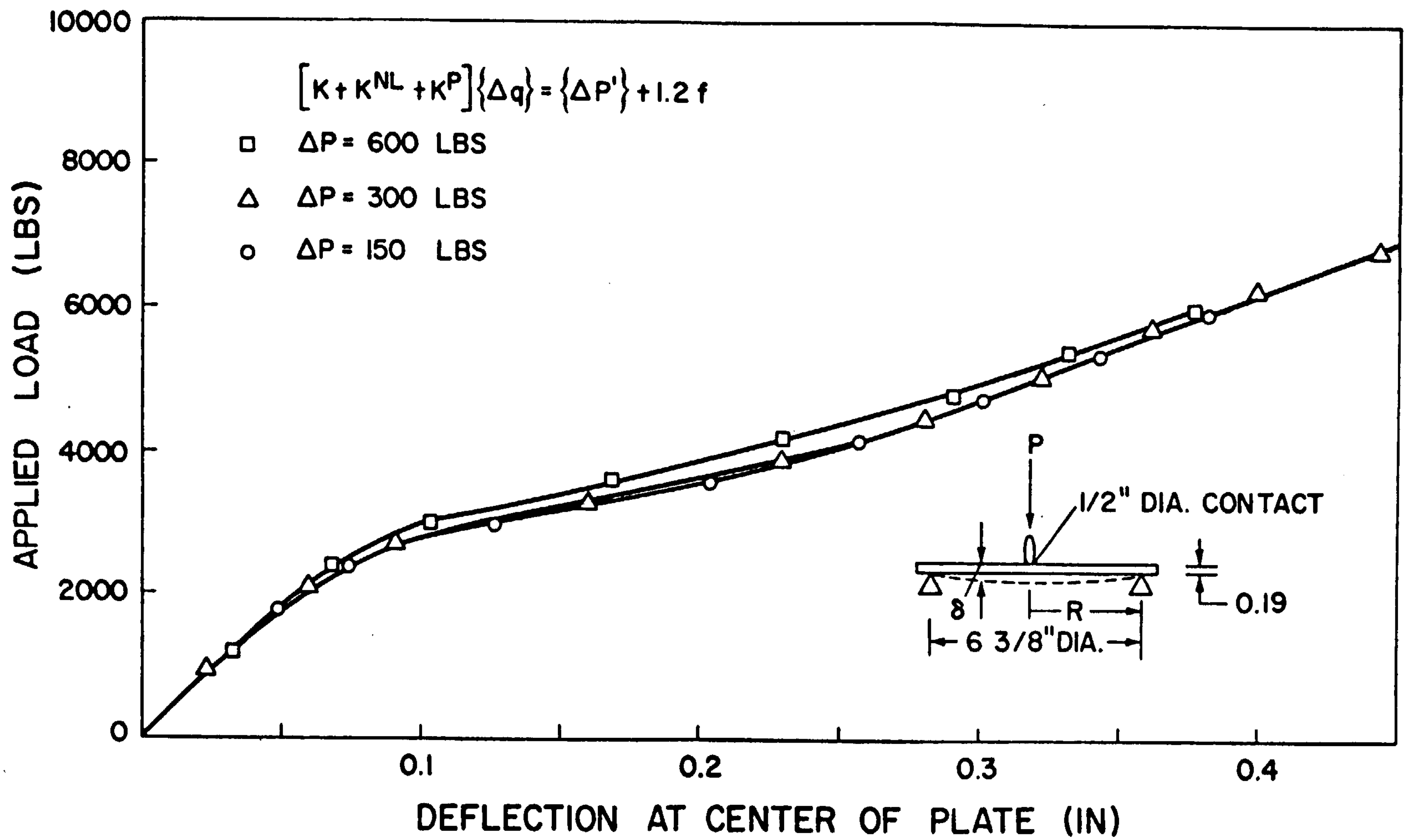

FIG. 16 CONVERGENCE STUDIES WITH LOAD REFINEMENT

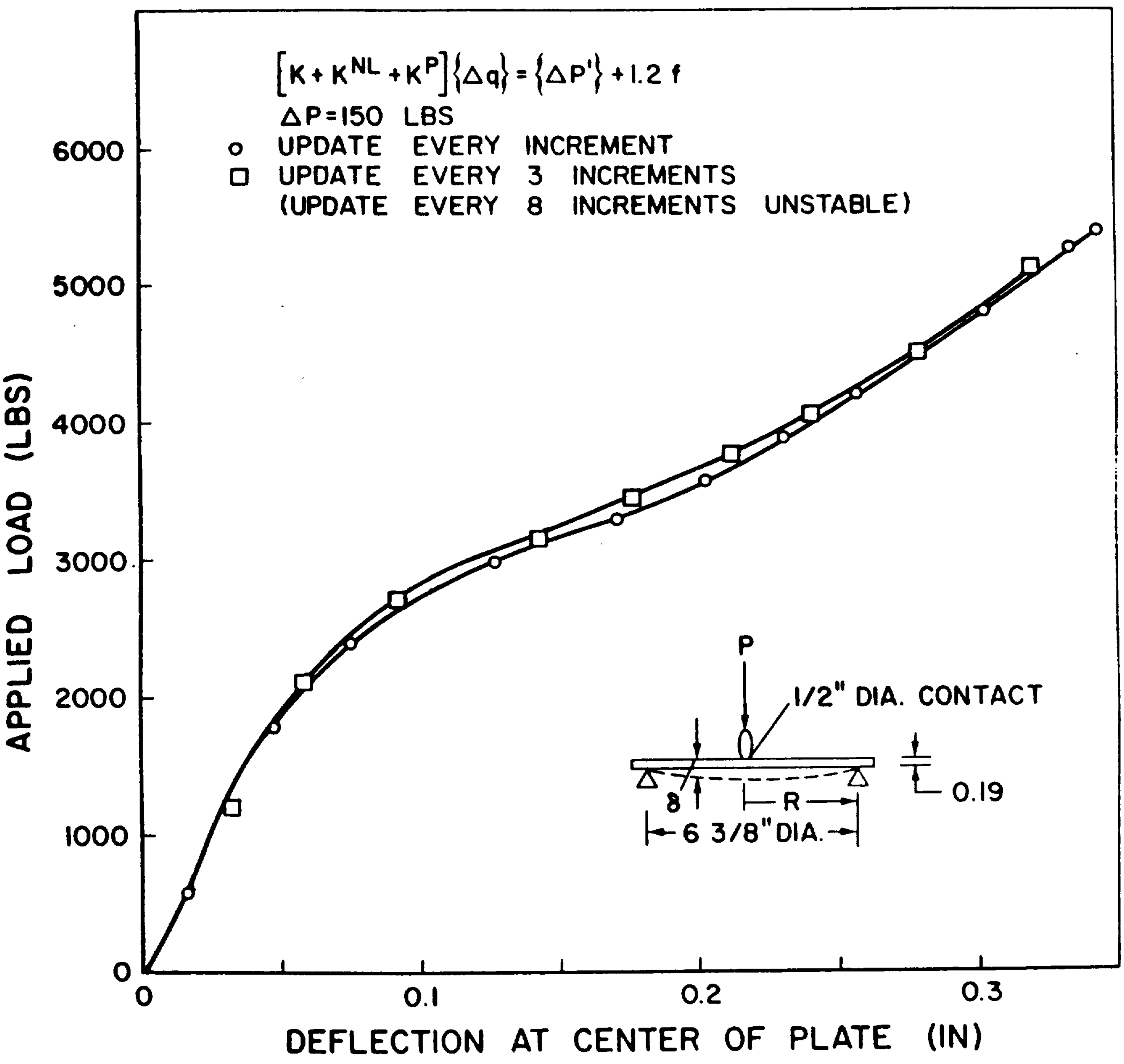

FIG. 17 EFFECT OF UPDATING STIFFNESS MATRIX ON LOAD-DEFLECTION CURVE

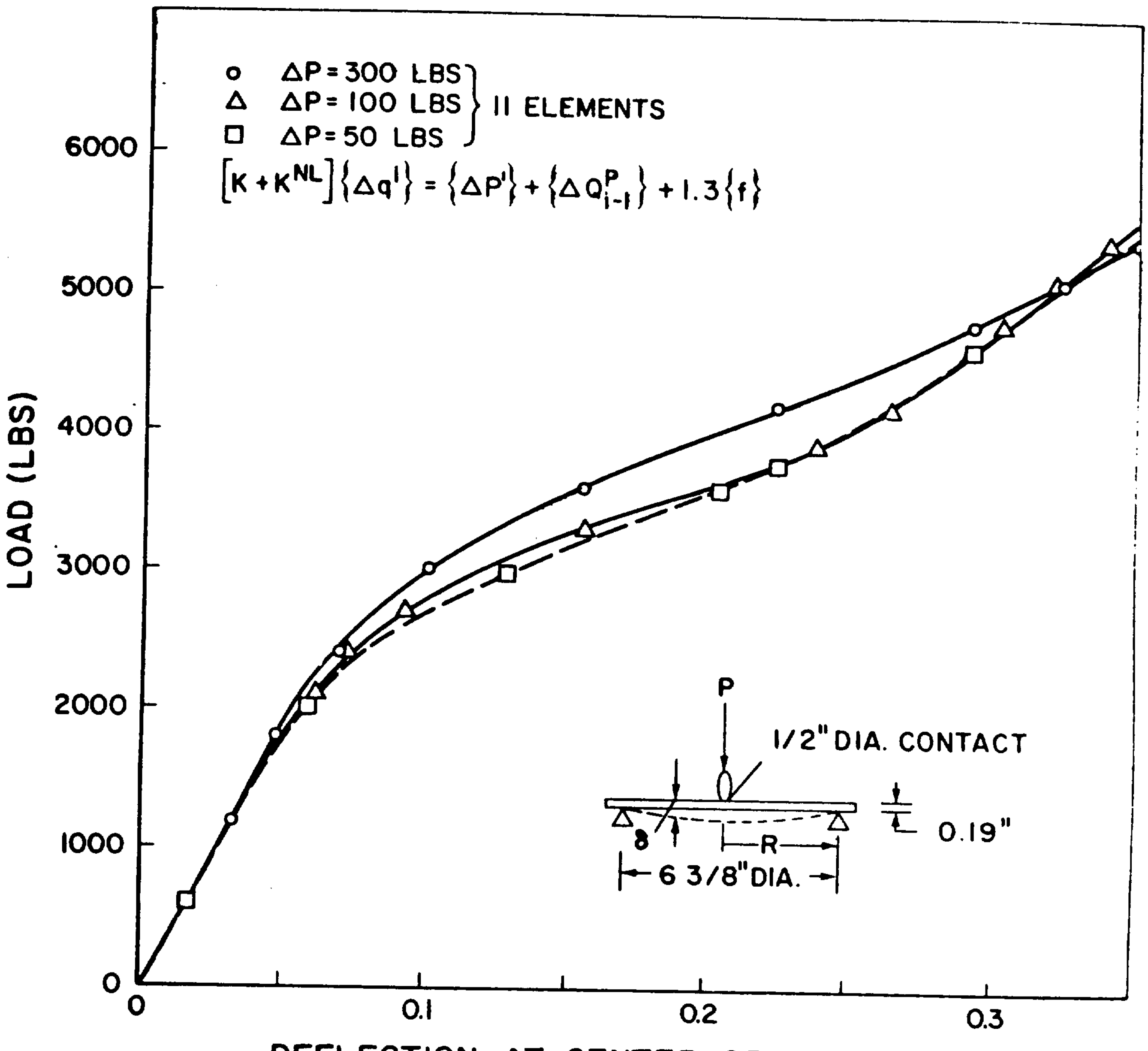

FIG.18 LOAD-DEFLECTION CURVES FOR

$(K + K^{NL})\Delta q^i = \Delta P^i + \Delta Q^P_{i-1} + 1.3f$

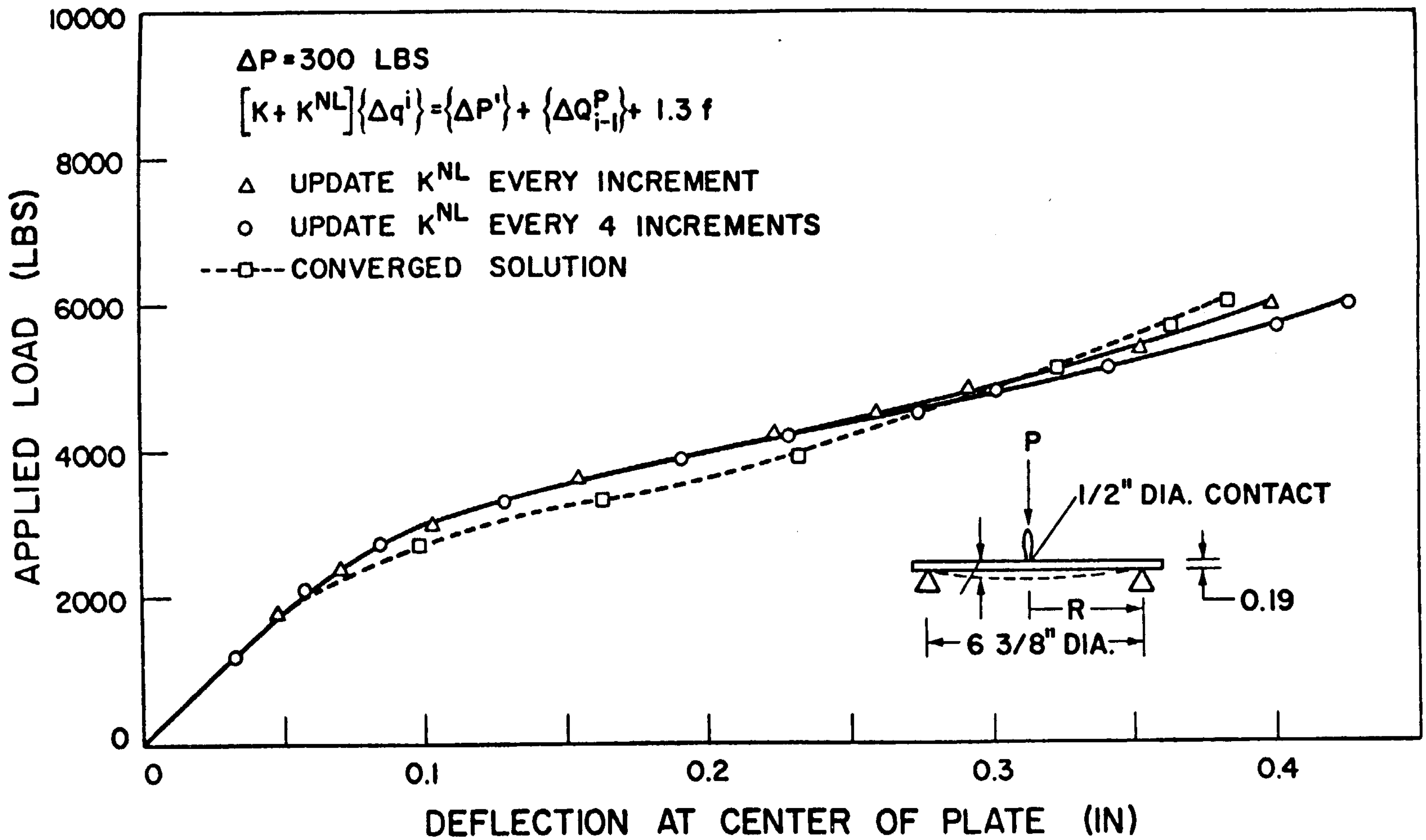

FIG. 19 EFFECTS OF UPDATING K^{NL} ON ACCURACY OF LOAD DEFLECTION CURVE

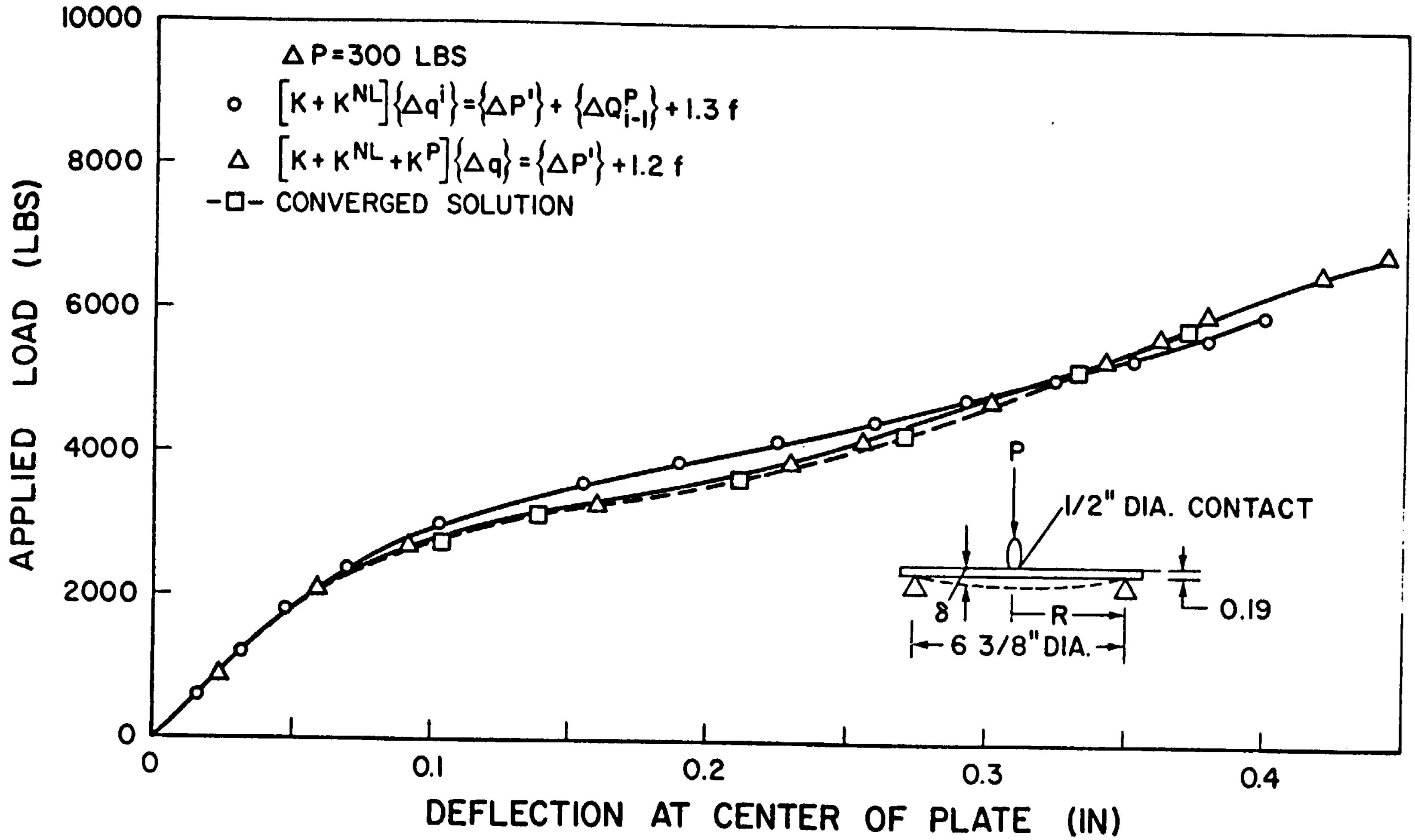

FIG. 20 EFFECTS OF REPRESENTATION OF PLASTICITY TERM ON THE ACCURACY OF LOAD-DEFLECTION CURVE

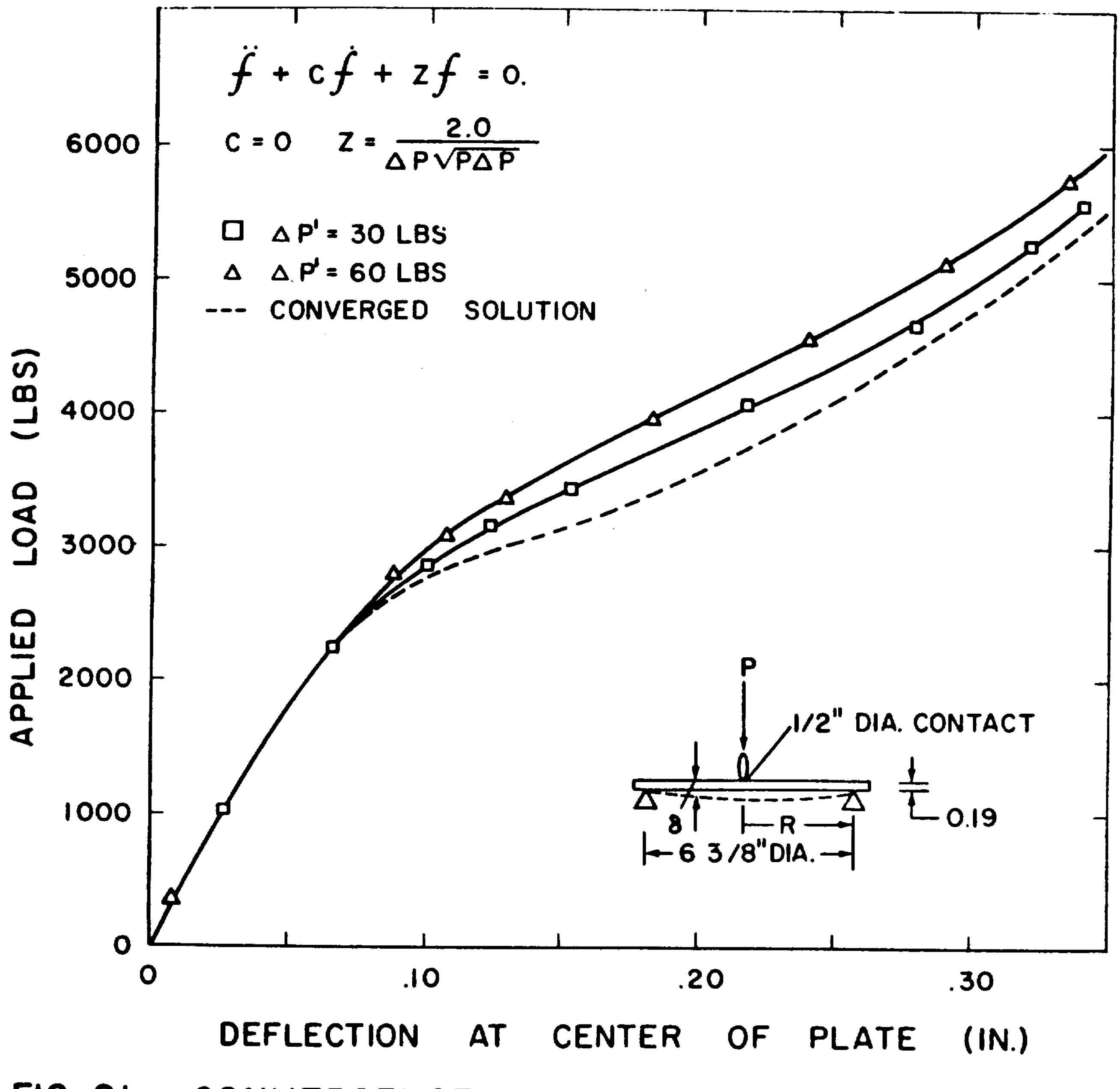

FIG. 21 CONVERGENCE OF LOAD – DEFLECTION CURVE FOR $\ddot{f} + C\dot{f} + Zf = 0$.

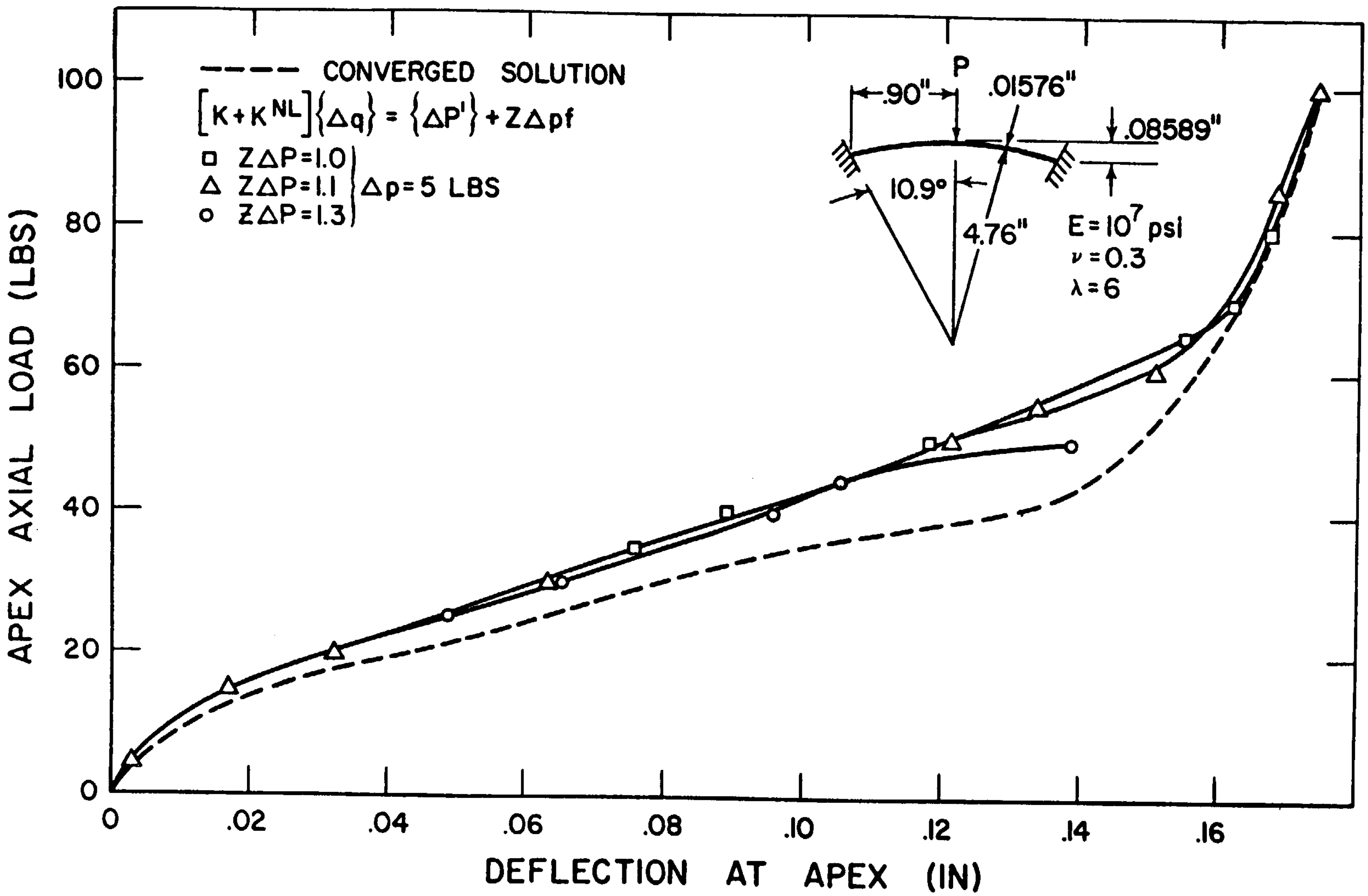

FIG. 22 ELASTIC LOAD-DEFLECTION CURVE FOR SHALLOW CAP

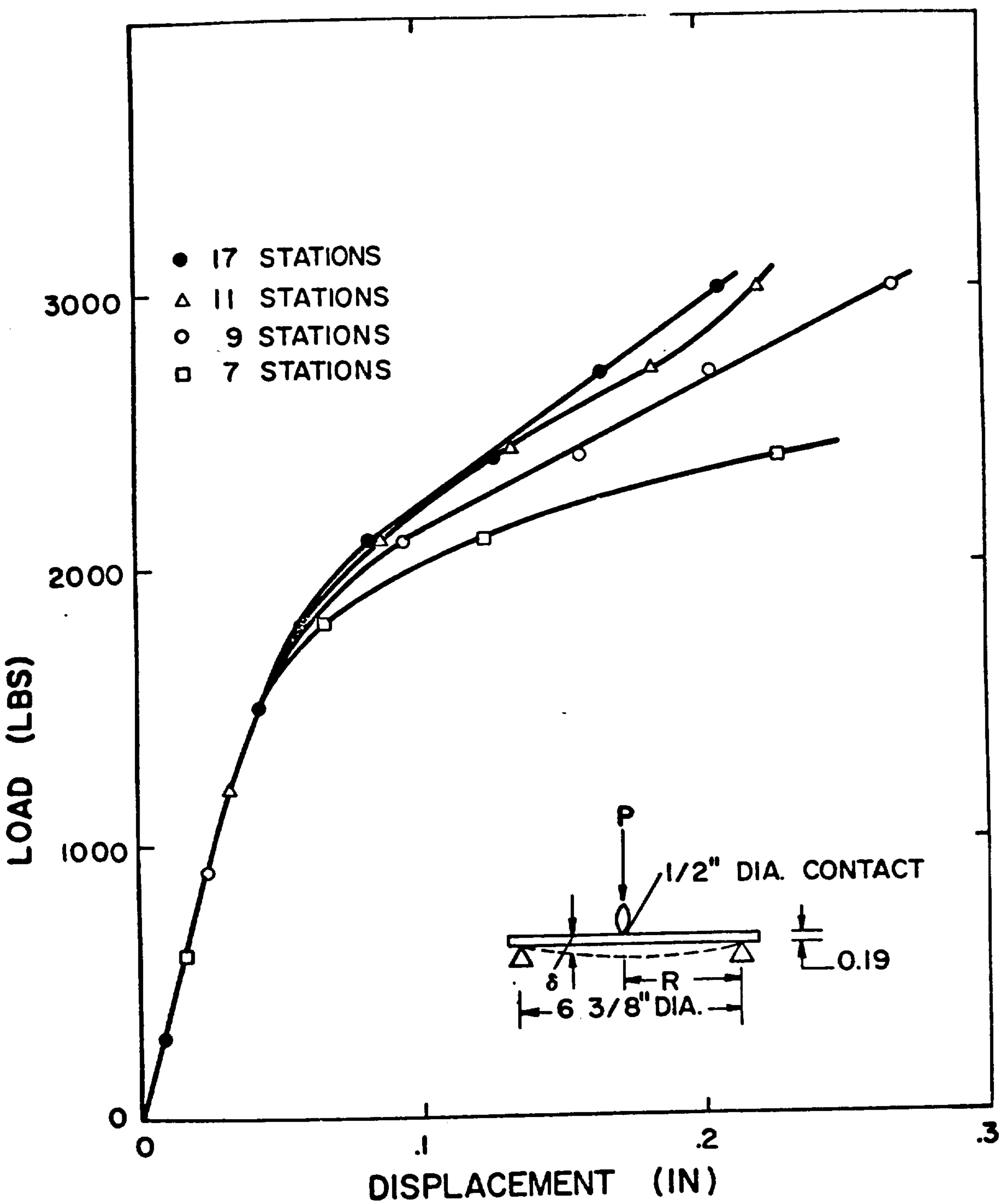

FIG. 23 CONVERGENCE OF TRAPEZOIDAL INTEGRATION THROUGH THICKNESS

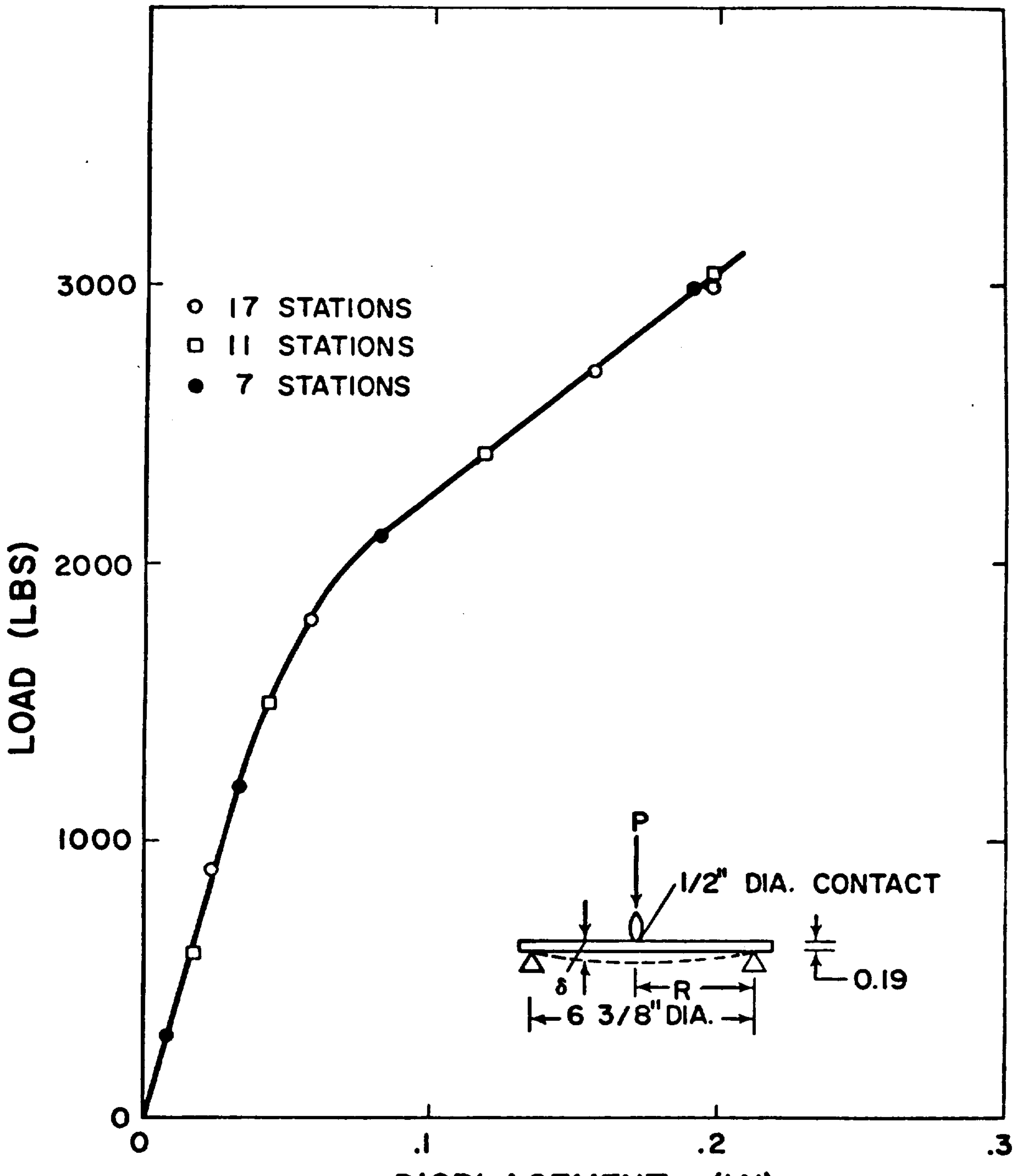

FIG. 24 CONVERGENCE OF SIMPSON INTEGRATION THROUGH THICKNESS

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